Ae/APh 101

Skeleton Guide

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Contents

1 Fundamentals

1.1 Control Volume Statements

Ω is a material volume, V is an arbitrary control volume, $\partial\Omega$ indicates the surface of the volume.
mass conservation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\rho\,\mathrm{d}V = 0 \tag{1}$$

Momentum conservation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\rho\mathbf{u}\,\mathrm{d}V = \mathbf{F} \tag{2}$$

Forces:

$$\mathbf{F} = \int_{\Omega}\rho\mathbf{G}\,\mathrm{d}V + \int_{\partial\Omega}\mathbf{T}\,\mathrm{d}A \tag{3}$$

Surface traction forces

$$\mathbf{T} = -P\hat{\mathbf{n}} + \boldsymbol{\tau}\cdot\hat{\mathbf{n}} = \mathsf{T}\cdot\hat{\mathbf{n}} \tag{4}$$

Stress tensor T

$$\mathsf{T} = -P\mathsf{I} + \boldsymbol{\tau} \quad \text{or} \quad \mathsf{T}_{ik} = -P\delta_{ik} + \tau_{ik} \tag{5}$$

where I is the unit tensor, which in cartesian coordinates is

$$\mathsf{I} = \delta_{ik} \tag{6}$$

Viscous stress tensor, shear viscosity μ, bulk viscosity μ_v

$$\tau_{ik} = 2\mu\left(D_{ik} - \frac{1}{3}\delta_{ik}D_{jj}\right) + \mu_v\delta_{ik}D_{jj} \quad \text{implicit sum on j} \tag{7}$$

Deformation tensor

$$D_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) \quad \text{or} \quad \frac{1}{2}\left(\boldsymbol{\nabla}\mathbf{u} + \boldsymbol{\nabla}\mathbf{u}^{\mathsf{T}}\right) \tag{8}$$

Energy conservation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\rho\left(e + \frac{|\mathbf{u}|^2}{2}\right)\mathrm{d}V = \dot{Q} + \dot{W} \tag{9}$$

Work:

$$\dot{W} = \int_{\Omega}\rho\mathbf{G}\cdot\mathbf{u}\,\mathrm{d}V + \int_{\partial\Omega}\mathbf{T}\cdot\mathbf{u}\,\mathrm{d}A \tag{10}$$

Heat:

$$\dot{Q} = -\int_{\partial\Omega}\mathbf{q}\cdot\hat{\mathbf{n}}\,\mathrm{d}A \tag{11}$$

heat flux $\mathbf{q}$, thermal conductivity k and thermal radiation $\mathbf{q_r}$

$$\mathbf{q} = -k\boldsymbol{\nabla}T + \mathbf{q}_r \tag{12}$$

Entropy inequality (2nd Law of Thermodynamics):

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\rho s\,\mathrm{d}V \geq -\int_{\partial\Omega}\frac{\mathbf{q}\cdot\hat{\mathbf{n}}}{T}\mathrm{d}A \tag{13}$$

1.2 Reynolds Transport Theorem

The multi-dimensional analog of Leibniz's theorem:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \phi(\mathbf{x},t)\,\mathrm{d}V = \int_{V(t)} \frac{\partial \phi}{\partial t}\,\mathrm{d}V + \int_{\partial V} \phi \mathbf{u}_V \cdot \hat{\mathbf{n}}\,\mathrm{d}A \tag{14}$$

The transport theorem proper. Material volume Ω, arbitrary volume V.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} \phi\,\mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t}\int_{V} \phi\,\mathrm{d}V + \int_{\partial V} \phi(\mathbf{u}-\mathbf{u}_V) \cdot \hat{\mathbf{n}}\,\mathrm{d}A \tag{15}$$

1.3 Integral Equations

The equations of motions can be rewritten with Reynolds Transport Theorem to apply to an (almost) arbitrary moving control volume. Beware of noninertial reference frames and the apparent forces or accelerations that such systems will introduce.

Moving control volume:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V} \rho \mathrm{d}V + \int_{\partial V} \rho\,(\mathbf{u}-\mathbf{u}_V) \cdot \hat{\mathbf{n}}\,\mathrm{d}A = 0 \tag{16}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V} \rho \mathbf{u} \mathrm{d}V + \int_{\partial V} \rho \mathbf{u}\,(\mathbf{u}-\mathbf{u}_V) \cdot \hat{\mathbf{n}}\,\mathrm{d}A = \int_{V} \rho \mathbf{G}\,\mathrm{d}V + \int_{\partial V} \mathbf{T}\,\mathrm{d}A \tag{17}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V} \rho \left(e + \frac{|\mathbf{u}|^2}{2}\right)\mathrm{d}V + \int_{\partial V} \rho \left(e + \frac{|\mathbf{u}|^2}{2}\right)(\mathbf{u}-\mathbf{u}_V) \cdot \hat{\mathbf{n}}\,\mathrm{d}A = \int_{V} \rho \mathbf{G}\cdot\mathbf{u}\,\mathrm{d}V + \int_{\partial V} \mathbf{T}\cdot\mathbf{u}\,\mathrm{d}A - \int_{\partial V} \mathbf{q}\cdot\hat{\mathbf{n}}\,\mathrm{d}A \tag{18}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V} \rho s \mathrm{d}V + \int_{\partial V} \rho s\,(\mathbf{u}-\mathbf{u}_V) \cdot \hat{\mathbf{n}}\,\mathrm{d}A + \int_{\partial V} \frac{\mathbf{q}}{T}\cdot\hat{\mathbf{n}}\,\mathrm{d}A \geq 0 \tag{19}$$

Stationary control volume:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V} \rho \mathrm{d}V + \int_{\partial V} \rho \mathbf{u} \cdot \hat{\mathbf{n}}\,\mathrm{d}A = 0 \tag{20}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V} \rho \mathbf{u} \mathrm{d}V + \int_{\partial V} \rho \mathbf{u}\mathbf{u} \cdot \hat{\mathbf{n}}\,\mathrm{d}A = \int_{V} \rho \mathbf{G}\,\mathrm{d}V + \int_{\partial V} \mathbf{T}\,\mathrm{d}A \tag{21}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V} \rho \left(e + \frac{|\mathbf{u}|^2}{2}\right)\mathrm{d}V + \int_{\partial V} \rho \left(e + \frac{|\mathbf{u}|^2}{2}\right)\mathbf{u} \cdot \hat{\mathbf{n}}\,\mathrm{d}A = \int_{V} \rho \mathbf{G}\cdot\mathbf{u}\,\mathrm{d}V + \int_{\partial V} \mathbf{T}\cdot\mathbf{u}\,\mathrm{d}A - \int_{\partial V} \mathbf{q}\cdot\hat{\mathbf{n}}\,\mathrm{d}A \tag{22}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V} \rho s \mathrm{d}V + \int_{\partial V} \rho s \mathbf{u} \cdot \hat{\mathbf{n}}\,\mathrm{d}A + \int_{\partial V} \frac{\mathbf{q}}{T}\cdot\hat{\mathbf{n}}\,\mathrm{d}A \geq 0 \tag{23}$$

1.3.1 Simple Control Volumes

Consider a stationary control volume V with $i = 1, 2, \ldots, I$ connections or openings through which there is fluid flowing in and $j = 1, 2, \ldots, J$ connections through which the fluid is following out. At the inflow and outflow stations, further suppose that we can define average or effective uniform properties h_i, ρ_i, u_i of the fluid. Then the mass conservation equation is

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho \mathrm{d}V = \sum_{i=1}^{I} A_i \dot{m}_i - \sum_{j=1}^{J} A_j \dot{m}_j \tag{24}$$

where A_i is the cross-sectional area of the ith connection and $\dot{m}_i = \rho_i u_i$ is the mass flow rate per unit area through this connection. The energy equation for this same situation is

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\left(e + \frac{|\mathbf{u}|^2}{2} + gz\right)\mathrm{d}V \quad &= \quad \sum_{i=1}^{I} A_i \dot{m}_i \left(h_i + \frac{|\mathbf{u}_i|^2}{2} + gz_i\right) \\ &\quad - \sum_{j=1}^{J} A_j \dot{m}_j \left(h_j + \frac{|\mathbf{u}_j|^2}{2} + gz_j\right) + \dot{Q} + \dot{W}\end{aligned} \tag{25}$$

where $\dot{Q}$ is the thermal energy (heat) transferred into the control volume and $\dot{W}$ is the mechanical work done on the fluid inside the control volume.

1.3.2 Steady Momentum Balance

For a stationary control volume, the steady momentum equation can be written as

$$\int_{\partial V} \rho \mathbf{u}\mathbf{u} \cdot \hat{\mathbf{n}}\, \mathrm{d}A + \int_{\partial V} P\hat{\mathbf{n}}\, \mathrm{d}A = \int_V \rho \mathbf{G}\, \mathrm{d}V + \int_{\partial V} \boldsymbol{\tau} \cdot \hat{\mathbf{n}}\, \mathrm{d}A + \mathbf{F}_{ext} \tag{26}$$

where $\mathbf{F}_{ext}$ are the external forces required to keep objects in contact with the flow in force equilibrium. These *reaction* forces are only needed if the control volume includes stationary objects or surfaces. For a control volume completely within the fluid, $\mathbf{F}_{ext} = 0$.

1.4 Vector Calculus

1.4.1 Vector Identities

If $\mathbf{A}$ and $\mathbf{B}$ are two differentiable vector fields $\mathbf{A}(\mathbf{x})$, $\mathbf{B}(\mathbf{x})$ and ϕ is a differentiable scalar field $\phi(\mathbf{x})$, then the following identities hold:

$$\boldsymbol{\nabla} \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{A} - (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B} - (\boldsymbol{\nabla} \cdot \mathbf{A})\mathbf{B} + (\boldsymbol{\nabla} \cdot \mathbf{B})\mathbf{A} \tag{27}$$

$$\boldsymbol{\nabla}(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{A} + (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B} + \mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{A}) + \mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{B}) \tag{28}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\phi) = 0 \tag{29}$$

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) = 0 \tag{30}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \tag{31}$$

$$\boldsymbol{\nabla} \times (\phi \mathbf{A}) = \boldsymbol{\nabla}\phi \times \mathbf{A} + \phi \boldsymbol{\nabla} \times \mathbf{A} \tag{32}$$

1.4.2 Curvilinear Coordinates

Scale factors Consider an orthogonal curvilinear coordinate system (x_1, x_2, x_3) defined by a triad of unit vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, which satisfy the orthogonality condition:

$$\mathbf{e}_i \cdot \mathbf{e}_k = \delta_{ik} \tag{33}$$

and form a right-handed coordinate system

$$\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2 \tag{34}$$

The *scale factors* h_i are defined by

$$\mathrm{d}\mathbf{r} = h_1 \mathrm{d}x_1 \mathbf{e}_1 + h_2 \mathrm{d}x_2 \mathbf{e}_2 + h_3 \mathrm{d}x_3 \mathbf{e}_3 \tag{35}$$

or

$$h_i \equiv \left| \frac{\partial \mathbf{r}}{\partial x_i} \right| \tag{36}$$

The unit of arc length in this coordinate system is $\mathrm{d}s^2 = \mathrm{d}\mathbf{r} \cdot \mathrm{d}\mathbf{r}$:

$$\mathrm{d}s^2 = h_1^2\,\mathrm{d}x_1^2 + h_2^2\,\mathrm{d}x_2^2 + h_3^2\,\mathrm{d}x_3^2 \tag{37}$$

The unit of differential volume is

$$\mathrm{d}V = h_1 h_2 h_3\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 \tag{38}$$

1.4.3 Gauss' Divergence Theorem

For a vector or tensor field $\mathbf{F}$, the following relationship holds:

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{F}\,\mathrm{d}V \equiv \int_{\partial V} \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}A \tag{39}$$

This leads to the simple interpretation of the divergence as the following limit

$$\boldsymbol{\nabla} \cdot \mathbf{F} \equiv \lim_{V \to 0} \frac{1}{V} \int_{\partial V} \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}A \tag{40}$$

A useful variation on the divergence theorem is

$$\int_V (\boldsymbol{\nabla} \times \mathbf{F})\,\mathrm{d}V \equiv \int_{\partial V} \hat{\mathbf{n}} \times \mathbf{F}\,\mathrm{d}A \tag{41}$$

This leads to the simple interpretation of the curl as

$$\boldsymbol{\nabla} \times \mathbf{F} \equiv \lim_{V \to 0} \frac{1}{V} \int_{\partial V} \hat{\mathbf{n}} \times \mathbf{F}\,\mathrm{d}A \tag{42}$$

1.4.4 Stokes' Theorem

For a vector or tensor field $\mathbf{F}$, the following relationship holds on an open, two-sided surface S bounded by a closed, non-intersecting curve ∂S:

$$\int_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, \mathrm{d}A \equiv \int_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \tag{43}$$

1.4.5 Div, Grad and Curl

The gradient operator $\boldsymbol{\nabla}$ or *grad* for a scalar field ψ is

$$\boldsymbol{\nabla}\psi = \frac{1}{h_1}\frac{\partial \psi}{\partial x_1}\,\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial \psi}{\partial x_2}\,\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial \psi}{\partial x_3}\mathbf{e}_3 \tag{44}$$

A simple interpretation of the gradient operator is in terms of the differential of a function in a direction $\hat{\mathbf{a}}$

$$\mathrm{d}_{\hat{\mathbf{a}}}\psi = \lim_{\mathrm{d}\mathbf{a}\to 0} \psi(\mathbf{x} + \mathrm{d}\mathbf{a}) - \psi(\mathbf{x}) = \boldsymbol{\nabla}\psi \cdot \mathrm{d}\mathbf{a} \tag{45}$$

The divergence operator $\boldsymbol{\nabla}\cdot$ or *div* is

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}(h_2 h_3 F_1) + \frac{\partial}{\partial x_2}(h_3 h_1 F_2) + \frac{\partial}{\partial x_3}(h_1 h_2 F_3)\right] \tag{46}$$

The curl operator $\boldsymbol{\nabla}\times$ or *curl* is

$$\boldsymbol{\nabla} \times \mathbf{F} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\,\mathbf{e}_1 & h_2\,\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix} \tag{47}$$

The components of the curl are:

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{F} \;&=\; \frac{\mathbf{e}_1}{h_2 h_3}\left[\frac{\partial}{\partial x_2}(h_3 F_3) - \frac{\partial}{\partial x_3}(h_2 F_2)\right] \\ &\quad + \frac{\mathbf{e}_2}{h_3 h_1}\left[\frac{\partial}{\partial x_3}(h_1 F_1) - \frac{\partial}{\partial x_1}(h_3 F_3)\right] \\ &\quad + \frac{\mathbf{e}_3}{h_1 h_2}\left[\frac{\partial}{\partial x_1}(h_2 F_2) - \frac{\partial}{\partial x_2}(h_1 F_1)\right] \end{aligned} \tag{48}$$

The Laplacian operator ∇^2 for a scalar field ψ is

$$\nabla^2\psi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \psi}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial \psi}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \psi}{\partial x_3}\right)\right] \tag{49}$$

1.4.6 Specific Coordinates

(x_1, x_2, x_3)	x	y	z	h_1	h_2	h_3
Cartesian						
(x, y, z)	x	y	z	1	1	1
Cylindrical						
(r, θ, z)	$r \sin\theta$	$r \cos\theta$	z	1	r	1
Spherical						
(r, ϕ, θ)	$r \sin\phi \cos\theta$	$r \sin\phi \sin\theta$	$r \cos\phi$	1	r	$r \sin\phi$
Parabolic Cylindrical						
(u, v, z)	$\frac{1}{2}(u^2 - v^2)$	uv	z	$\sqrt{u^2 + v^2}$	h_1	1
Paraboloidal						
(u, v, ϕ)	$uv \cos\phi$	$uv \sin\phi$	$\frac{1}{2}(u^2 - v^2)$	$\sqrt{u^2 + v^2}$	h_1	uv
Elliptic Cylindrical						
(u, v, z)	$a \cosh u \cos v$	$a \sinh u \sin v$	z	$a\sqrt{\sinh^2 u + \sin^2 v}$	h_1	1
Prolate Spheroidal						
(ξ, η, ϕ)	$a \sinh\xi \sin\eta \cos\phi$	$a \sinh\xi \sin\eta \sin\phi$	$a \cosh\xi \cos\eta$	$a\sqrt{\sinh^2 \xi + \sin^2 \eta}$	h_1	$a \sinh\xi \sin\eta$

1.5 Differential Relations

1.5.1 Conservation form

The equations are first written in conservation form

$$\frac{\partial}{\partial t} \text{density} + \boldsymbol{\nabla} \cdot \text{flux} = \text{source} \tag{50}$$

for a fixed (Eulerian) control volume in an inertial reference frame by using the divergence theorem.

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0 \tag{51}$$

$$\frac{\partial}{\partial t} (\rho \mathbf{u}) + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}\mathbf{u} - \mathsf{T}) = \rho \mathbf{G} \tag{52}$$

$$\frac{\partial}{\partial t} \rho \left(e + \frac{|\mathbf{u}|^2}{2} \right) + \boldsymbol{\nabla} \cdot \left[\rho \mathbf{u} \left(e + \frac{|\mathbf{u}|^2}{2} \right) - \mathsf{T} \cdot \mathbf{u} + \mathbf{q} \right] = \rho \mathbf{G} \cdot \mathbf{u} \tag{53}$$

$$\frac{\partial}{\partial t} (\rho s) + \boldsymbol{\nabla} \cdot \left(\rho \mathbf{u} s + \frac{\mathbf{q}}{T} \right) \geq 0 \tag{54}$$

1.6 Convective Form

This form uses the *convective* or *material* derivative

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \tag{55}$$

$$\begin{aligned}
\frac{\mathrm{D}\rho}{\mathrm{D}t} &= -\rho\boldsymbol{\nabla}\cdot\mathbf{u} && (56)\\
\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} &= -\boldsymbol{\nabla}P + \boldsymbol{\nabla}\cdot\boldsymbol{\tau} + \rho\mathbf{G} && (57)\\
\rho\frac{\mathrm{D}}{\mathrm{D}t}\left(e + \frac{|\mathbf{u}|^2}{2}\right) &= \boldsymbol{\nabla}\cdot(\mathsf{T}\cdot\mathbf{u}) - \boldsymbol{\nabla}\cdot\mathbf{q} + \rho\mathbf{G}\cdot\mathbf{u} && (58)\\
\rho\frac{\mathrm{D}s}{\mathrm{D}t} &\geq -\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{q}}{T}\right) && (59)
\end{aligned}$$

Alternate forms of the energy equation:

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}\left(e + \frac{|\mathbf{u}|^2}{2}\right) = -\boldsymbol{\nabla}\cdot(P\mathbf{u}) + \boldsymbol{\nabla}\cdot(\boldsymbol{\tau}\cdot\mathbf{u}) - \boldsymbol{\nabla}\cdot\mathbf{q} + \rho\mathbf{G}\cdot\mathbf{u} \tag{60}$$

Formulation using enthalpy $h = e + P/\rho$

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}\left(h + \frac{|\mathbf{u}|^2}{2}\right) = \frac{\partial P}{\partial t} + \boldsymbol{\nabla}\cdot(\boldsymbol{\tau}\cdot\mathbf{u}) - \boldsymbol{\nabla}\cdot\mathbf{q} + \rho\mathbf{G}\cdot\mathbf{u} \tag{61}$$

Mechanical energy equation

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}\frac{|\mathbf{u}|^2}{2} = -\left(\mathbf{u}\cdot\boldsymbol{\nabla}\right)P + \mathbf{u}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\tau} + \rho\mathbf{G}\cdot\mathbf{u} \tag{62}$$

Thermal energy equation

$$\frac{\mathrm{D}e}{\mathrm{D}t} = -P\frac{\mathrm{D}v}{\mathrm{D}t} + v\boldsymbol{\tau}{:}\boldsymbol{\nabla}\mathbf{u} - v\boldsymbol{\nabla}\cdot\mathbf{q} \tag{63}$$

Dissipation

$$\Upsilon = \boldsymbol{\tau}{:}\boldsymbol{\nabla}\mathbf{u} = \tau_{ik}\frac{\partial u_i}{\partial x_k} \quad \text{sum on } i \text{ and } k \tag{64}$$

Entropy

$$\rho\frac{\mathrm{D}s}{\mathrm{D}t} = -\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{q}}{T}\right) + \frac{\Upsilon}{T} + k\left(\frac{\boldsymbol{\nabla}T}{T}\right)^2 \tag{65}$$

1.7 Divergence of Viscous Stress

For a fluid with constant μ and μ_v, the divergence of the viscous stress in Cartesian coordinates can be reduced to:

$$\boldsymbol{\nabla}\cdot\boldsymbol{\tau} = \mu\nabla^2\mathbf{u} + \left(\mu_v + \frac{1}{3}\mu\right)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) \tag{66}$$

1.8 Euler Equations

Inviscid, no heat transfer, no body forces.

$$\begin{aligned} \frac{\mathrm{D}\rho}{\mathrm{D}t} &= -\rho\boldsymbol{\nabla}\cdot\mathbf{u} && (67)\\ \rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} &= -\boldsymbol{\nabla}P && (68)\\ \rho\frac{\mathrm{D}}{\mathrm{D}t}\left(h+\frac{|\mathbf{u}|^2}{2}\right) &= \frac{\partial P}{\partial t} && (69)\\ \frac{\mathrm{D}s}{\mathrm{D}t} &\geq 0 && (70) \end{aligned}$$

1.9 Bernoulli Equation

Consider the unsteady energy equation in the form

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}\left(h+\frac{|\mathbf{u}|^2}{2}\right) = \frac{\partial P}{\partial t} + \boldsymbol{\nabla}\cdot(\boldsymbol{\tau}\cdot\mathbf{u}) - \boldsymbol{\nabla}\cdot\mathbf{q} + \rho\mathbf{G}\cdot\mathbf{u} \tag{71}$$

and further suppose that the external force field $\mathbf{G}$ is *conservative* and can be derived from a potential Φ as

$$\mathbf{G} = -\boldsymbol{\nabla}\Phi \tag{72}$$

then if $\Phi(\mathbf{x})$ only, we have

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}\left(h+\frac{|\mathbf{u}|^2}{2}+\Phi\right) = \frac{\partial P}{\partial t} + \boldsymbol{\nabla}\cdot(\boldsymbol{\tau}\cdot\mathbf{u}) - \boldsymbol{\nabla}\cdot\mathbf{q} \tag{73}$$

The Bernoulli constant is

$$H = h + \frac{|\mathbf{u}|^2}{2} + \Phi \tag{74}$$

In the absence of unsteadiness, viscous forces and heat transfer we have

$$\mathbf{u}\cdot\boldsymbol{\nabla}\left(h+\frac{|\mathbf{u}|^2}{2}+\Phi\right) = 0 \tag{75}$$

Or

$$H_\circ = \text{constant on streamlines}$$

For the ordinary case of isentropic flow of an incompressible fluid $\mathrm{d}h = \mathrm{d}P/\rho_\circ$ in a uniform gravitational field $\Phi = g(z - z_\circ)$, we have the standard result

$$P + \rho_\circ\frac{|\mathbf{u}|^2}{2} + \rho_\circ g z = \text{constant} \tag{76}$$

1.10 Vorticity

Vorticity is defined as

$$\boldsymbol{\omega} \equiv \boldsymbol{\nabla} \times \mathbf{u} \tag{77}$$

and the vector identities can be used to obtain

$$(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = \boldsymbol{\nabla}(\frac{|\mathbf{u}|^2}{2}) - \mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{u}) \tag{78}$$

The momentum equation can be reformulated to read:

$$\boldsymbol{\nabla} H = \boldsymbol{\nabla}\left(h + \frac{|\mathbf{u}|^2}{2} + \Phi\right) = -\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \times \boldsymbol{\omega} + T\boldsymbol{\nabla} s + \frac{\boldsymbol{\nabla} \cdot \boldsymbol{\tau}}{\rho} \tag{79}$$

1.11 Dimensional Analysis

Fundamental Dimensions

L	length	meter (m)
M	mass	kilogram (kg)
T	time	second (s)
θ	temperature	Kelvin (K)
I	current	Ampere (A)

Some derived dimensional units

force	Newton (N)	MLT^{-2}
pressure	Pascal (Pa)	$ML^{-1}T^{-2}$
	bar = 10^5 Pa	
energy	Joule (J)	ML^2T^{-2}
frequency	Hertz (Hz)	T^{-1}
power	Watt (W)	ML^2T^{-3}
viscosity (μ)	Poise (P)	$ML^{-1}T^{-1}$

Pi Theorem Given n dimensional variables X_1, X_2, …, X_n, and f independent fundamental dimensions (at most 5) involved in the problem:

1. The number of dimensionally independent variables r is

$$r \leq f$$

2. The number $p = n$ - r of dimensionless variables Π_i

$$\Pi_i = \frac{X_i}{X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_r^{\alpha_r}}$$

that can be formed is

$$p \geq n - f$$

Conventional Dimensionless Numbers

Reynolds	Re	$\rho UL/\mu$
Mach	Ma	U/c
Prandtl	Pr	$\mu c_P/k = \nu/\kappa$
Strouhal	St	L/UT
Knudsen	Kn	Λ/L
Peclet	Pe	UL/κ
Schmidt	Sc	ν/D
Lewis	Le	D/κ

Reference conditions: U, velocity; μ, vicosity; D, mass diffusivity; k, thermal conductivity; L, length scale; T, time scale; c, sound speed; Λ, mean free path; c_P, specific heat at constant pressure.

Parameters for Air and Water Values given for nominal standard conditions 20 C and 1 bar.

			Air	Water
shear viscosity	μ	(kg/ms)	1.8×10^{-5}	1.00×10^{-3}
kinematic viscosity	ν	(m^2/s)	1.5×10^{-5}	1.0×10^{-6}
thermal conductivity	k	(W/mK)	2.54×10^{-2}	0.589
thermal diffusivity	κ	(m^2/s)	2.1×10^{-5}	1.4×10^{-7}
specific heat	c_p	(J/kgK)	1004.	4182.
sound speed	c	(m/s)	343.3	1484
density	ρ	(kg/m^3)	1.2	998.
gas constant	R	(m^2/s^2K)	287	462.
thermal expansion	β	(K^{-1})	3.3×10^{-4}	2.1×10^{-4}
isentropic compressibility	κ_s	(Pa^{-1})	7.01×10^{-6}	4.5×10^{-10}
Prandtl number	Pr		.72	7.1
Fundamental derivative	Γ		1.205	4.4
ratio of specific heats	γ		1.4	1.007
Grüneisen coefficient	G		0.40	0.11

2 Thermodynamics

2.1 Thermodynamic potentials and fundamental relations

$$\begin{aligned}
\textbf{energy} \quad e(s,v) & \\
\mathrm{d}e &= T\,\mathrm{d}s - P\,\mathrm{d}v && (80) \\
\textbf{enthalpy} \quad h(s,P) &= e + Pv \\
\mathrm{d}h &= T\,\mathrm{d}s + v\,\mathrm{d}P && (81) \\
\textbf{Helmholtz} \quad f(T,v) &= e - Ts \\
\mathrm{d}f &= -s\,\mathrm{d}T - P\,\mathrm{d}v && (82) \\
\textbf{Gibbs} \quad g(T,P) &= e - Ts + Pv \\
\mathrm{d}g &= -s\,\mathrm{d}T + v\,\mathrm{d}P && (83)
\end{aligned}$$

2.2 Maxwell relations

$$\left.\frac{\partial T}{\partial v}\right)_s = -\left.\frac{\partial P}{\partial s}\right)_v \qquad (84)$$

$$\left.\frac{\partial T}{\partial P}\right)_s = \left.\frac{\partial v}{\partial s}\right)_P \qquad (85)$$

$$\left.\frac{\partial s}{\partial v}\right)_T = \left.\frac{\partial P}{\partial T}\right)_v \qquad (86)$$

$$\left.\frac{\partial s}{\partial P}\right)_T = -\left.\frac{\partial v}{\partial T}\right)_P \qquad (87)$$

Calculus identities:

$$F(x,y,\dots) \qquad \mathrm{d}F = \left.\frac{\partial F}{\partial x}\right)_{y,z,\dots} \mathrm{d}x + \left.\frac{\partial F}{\partial y}\right)_{x,z,\dots} \mathrm{d}y + \dots \qquad (88)$$

$$\left.\frac{\partial x}{\partial y}\right)_f = -\frac{\left.\frac{\partial f}{\partial y}\right)_x}{\left.\frac{\partial f}{\partial x}\right)_y} \qquad (89)$$

$$\left.\frac{\partial x}{\partial f}\right)_y = \frac{1}{\left.\frac{\partial f}{\partial x}\right)_y} \qquad (90)$$

2.3 Various defined quantities

$$\begin{aligned}
\text{specific heat at constant volume} \quad c_v &\equiv \left.\frac{\partial e}{\partial T}\right)_v && (91)\\
\text{specific heat at constant pressure} \quad c_p &\equiv \left.\frac{\partial h}{\partial T}\right)_P && (92)\\
\text{ratio of specific heats} \quad \gamma &\equiv \frac{c_p}{c_v} && (93)\\
\text{sound speed} \quad c &\equiv \sqrt{\left.\frac{\partial P}{\partial \rho}\right)_s} && (94)\\
\text{coefficient of thermal expansion} \quad \beta &\equiv \frac{1}{v}\left.\frac{\partial v}{\partial T}\right)_P && (95)\\
\text{isothermal compressibility} \quad K_T &\equiv -\frac{1}{v}\left.\frac{\partial v}{\partial P}\right)_T && (96)\\
\text{isentropic compressibility} \quad K_s &\equiv -\frac{1}{v}\left.\frac{\partial v}{\partial P}\right)_s = \frac{1}{\rho c^2} && (97)
\end{aligned}$$

Specific heat relationships

$$K_T = \gamma K_s \quad \text{or} \quad \left.\frac{\partial P}{\partial v}\right)_s = \gamma \left.\frac{\partial P}{\partial v}\right)_T \tag{98}$$

$$c_p - c_v = -T\left(\frac{\partial P}{\partial v}\right)_T \left(\frac{\partial v}{\partial T}\right)_P^2 \tag{99}$$

Fundamental derivative

$$\begin{aligned}
\Gamma &\equiv \frac{c^4}{2v^3}\left.\frac{\partial^2 v}{\partial P^2}\right)_s && (100)\\
&= \frac{v^3}{2c^2}\left(\frac{\partial^2 P}{\partial v^2}\right)_s && (101)\\
&= 1 + \rho c\left(\frac{\partial c}{\partial P}\right)_s && (102)\\
&= \frac{1}{2}\left(\frac{v^2}{c^2}\left(\frac{\partial^2 h}{\partial v^2}\right)_s + 1\right) && (103)
\end{aligned}$$

Sound speed (squared)

$$\begin{aligned} c^2 &\equiv \left.\frac{\partial P}{\partial \rho}\right)_s && (104)\\ &= -v^2 \left.\frac{\partial P}{\partial v}\right)_s && (105)\\ &= \frac{v}{K_s} && (106)\\ &= \gamma \frac{v}{K_t} && (107) \end{aligned}$$

Grüneisen Coefficient

$$\begin{aligned} G &\equiv \frac{v\beta}{c_v K_T} && (108)\\ &= v\left(\frac{\partial P}{\partial e}\right)_v && (109)\\ &= \frac{v\beta}{c_p K_s} && (110)\\ &= -\frac{v}{T}\left(\frac{\partial T}{\partial v}\right)_s && (111) \end{aligned}$$

2.4 $v(P, s)$ relation

$$\begin{aligned} \frac{\mathrm{d}v}{v} &= -K_s\,\mathrm{d}P + \Gamma(K_s\,\mathrm{d}P)^2 + \beta\frac{T\,\mathrm{d}s}{c_p} + \ldots && (112)\\ &= -\frac{\mathrm{d}P}{\rho c^2} + \Gamma\left(\frac{\mathrm{d}P}{\rho c^2}\right)^2 + G\frac{T\mathrm{d}s}{c^2} + \ldots && (113) \end{aligned}$$

2.5 Equation of State Construction

Given $c_v(v, T)$ and $P(v, T)$, integrate

$$\begin{aligned} \mathrm{d}e &= c_v\,\mathrm{d}T + \left(T\left.\frac{\partial P}{\partial T}\right)_v - P\right)\mathrm{d}v && (114)\\ \mathrm{d}s &= \frac{c_v}{T}\,\mathrm{d}T + \left.\frac{\partial P}{\partial T}\right)_v \mathrm{d}v && (115) \end{aligned}$$

along two paths: I: variable T, fixed ρ and II: variable ρ, fixed T.

Energy:

$$e = e_\circ + \underbrace{\int_{T_\circ}^{T} c_v(T, \rho_\circ)\,\mathrm{d}T}_{\text{I}} + \underbrace{\int_{\rho_\circ}^{\rho}\left(P - T\left.\frac{\partial P}{\partial T}\right)_\rho\right)\frac{\mathrm{d}\rho}{\rho^2}}_{\text{II}} \quad (116)$$

Ideal gas limit $\rho_\circ \to 0$,

$$\lim_{\rho_\circ \to 0} c_v(T, \rho_\circ) = c_v^{ig}(T) \tag{117}$$

The ideal gas limit of I is the ideal gas internal energy

$$e^{ig}(T) = \int_{T_\circ}^{T} c_v^{ig}(T)\,\mathrm{d}T \tag{118}$$

Ideal gas limit of II is the *residual function*

$$e^r(\rho, T) = \int_0^\rho \left(P - T \left. \frac{\partial P}{\partial T} \right)_\rho \right) \frac{\mathrm{d}\rho}{\rho^2} \tag{119}$$

and the complete expression for internal energy is

$$e(\rho, T) = e_\circ + e^{ig}(T) + e^r(\rho, T) \tag{120}$$

Entropy:

$$s = s_\circ + \underbrace{\int_{T_\circ}^{T} \frac{c_v(T, \rho_\circ)}{T}\,\mathrm{d}T}_{\mathrm{I}} + \underbrace{\int_{\rho_\circ}^{\rho} \left(- \left. \frac{\partial P}{\partial T} \right)_\rho \right) \frac{\mathrm{d}\rho}{\rho^2}}_{\mathrm{II}} \tag{121}$$

The ideal gas limit $\rho_\circ \to 0$ has to be carried out slightly differently since the ideal gas entropy, unlike the internal energy, is a function of density and is singular at $\rho = 0$. Define

$$s^{ig} = \int_{T_\circ}^{T} \frac{c_v^{ig}(T)}{T}\,\mathrm{d}T - R \int_{\rho_\circ}^{\rho} \frac{\mathrm{d}\rho}{\rho} \tag{122}$$

where the second integral on the RHS is $R \ln \rho_\circ/\rho$. Then compute the residual function by substracting the singular part before carrying out the integration

$$s^r(\rho, T) = \int_0^\rho \left(R - \frac{1}{\rho} \left. \frac{\partial P}{\partial T} \right)_\rho \right) \frac{\mathrm{d}\rho}{\rho} \tag{123}$$

and the complete expression for entropy is

$$s(\rho, T) = s_\circ + s^{ig}(\rho, T) + s^r(\rho, T) \tag{124}$$

3 Compressible Flow

3.1 Steady Flow

A steady flow must be considered as compressible when the *Mach* number $M = u/c$ is sufficiently large. In an isentropic flow, the change in density produced by a speed u can be estimated as

$$\Delta\rho_s = c^{-2}\Delta P \sim -\frac{1}{2}\rho M^2 \tag{125}$$

from the energy equation discussed below and the fundamental relation of thermodynamics.

If the flow is unsteady, then the change in the density along the pathlines for inviscid flows without body forces is

$$\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\boldsymbol{\nabla}\cdot\mathbf{u} = -\frac{\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}^2}{2c^2} - \frac{1}{c^2}\left[\frac{1}{2}\frac{\partial u^2}{\partial t} - \frac{1}{\rho}\frac{\partial P}{\partial t}\right] \tag{126}$$

This first term is the steady flow condition $\sim M^2$. The second set of terms in the square braces are the unsteady contributions. These will be significant when the time scale T is comparable to the acoustic transit time $L/c_\circ$, i.e., $T \sim Lc_o$.

3.1.1 Streamlines and Total Properties

Stream lines $\mathbf{X}(t;\mathbf{x}_\circ)$ are defined by

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} = \mathbf{u} \qquad \mathbf{X} = \mathbf{x}_\circ \quad \text{when} \quad t = 0 \tag{127}$$

which in Cartesian coordinates yields

$$\frac{\mathrm{d}x_1}{u_1} = \frac{\mathrm{d}x_2}{u_2} = \frac{\mathrm{d}x_3}{u_3} \tag{128}$$

Total enthalpy is constant along streamlines in adiabatic, steady, inviscid flow

$$h_t = h + \frac{|\mathbf{u}|^2}{2} = \text{constant} \tag{129}$$

Velocity along a streamline is given by the energy equation:

$$u = |\mathbf{u}| = \sqrt{2(h_t - h)} \tag{130}$$

Total properties are defined in terms of total enthalpy and an idealized isentropic deceleration process along a streamline. Total pressure is defined by

$$P_t \equiv P(s_\circ, h_t) \tag{131}$$

Other total properties T_t, ρ_t, etc. can be computed from the equation of state.

3.2 Quasi-One Dimensional Flow

Adiabatic, frictionless flow:

$$\begin{aligned}
\mathrm{d}(\rho u A) &= 0 && (132)\\
\rho u \mathrm{d}u &= -dP && (133)\\
h + \frac{u^2}{2} &= \text{constant or} \qquad \mathrm{d}h = -u\mathrm{d}u && (134)\\
ds &\geq 0 && (135)
\end{aligned}$$

3.2.1 Isentropic Flow

If $\mathrm{d}s = 0$, then

$$\mathrm{d}P = c^2\mathrm{d}\rho + c^2(\Gamma - 1)\frac{(\mathrm{d}\rho)^2}{\rho} + \ldots \tag{136}$$

For isentropic flow, the quasi-one-dimensional equations can be written in terms of the Mach number as:

$$\begin{aligned}
\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}x} &= \frac{M^2}{1-M^2}\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}x} && (137)\\
\frac{1}{\rho c^2}\frac{\mathrm{d}P}{\mathrm{d}x} &= \frac{M^2}{1-M^2}\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}x} && (138)\\
\frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x} &= -\frac{1}{1-M^2}\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}x} && (139)\\
\frac{1}{M}\frac{\mathrm{d}M}{\mathrm{d}x} &= -\frac{1+(\Gamma-1)M^2}{1-M^2}\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}x} && (140)\\
\frac{1}{c^2}\frac{\mathrm{d}h}{\mathrm{d}x} &= \frac{M^2}{1-M^2}\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}x} && (141)
\end{aligned}$$

At a throat, the gradient in Mach number is:

$$\left(\frac{\mathrm{d}M}{\mathrm{d}x}\right)^2 = \frac{\Gamma}{2A}\frac{\mathrm{d}^2A}{\mathrm{d}x^2} \tag{142}$$

Constant-Γ Gas If the value of Γ is assumed to be constant, the quasi-one dimensional equations can be integrated to yield:

$$\frac{\rho_t}{\rho} = \left(1+(\Gamma-1)M^2\right)^{\frac{1}{2(\Gamma-1)}} \tag{143}$$

$$\frac{c_t}{c} = \left(\frac{\rho_t}{\rho}\right)^{\Gamma-1} = \left(1+(\Gamma-1)M^2\right)^{1/2} \tag{144}$$

$$\frac{h_t}{h} = \left(1+(\Gamma-1)M^2\right) \tag{145}$$

$$u = c_t\left(\frac{M^2}{1+(\Gamma-1)M^2}\right)^{1/2} \tag{146}$$

$$\frac{A}{A^*} = \frac{1}{M}\left(\frac{1+(\Gamma-1)M^2}{\Gamma}\right)^{\frac{\Gamma}{2(\Gamma-1)}} \tag{147}$$

$$\frac{P-P_t}{\rho_t c_t^2} = \frac{1}{2\Gamma-1}\left[\left(1+(\Gamma-1)M^2\right)^{-\frac{2\Gamma-1}{2(\Gamma-1)}}-1\right] \tag{148}$$

$$ \tag{149}$$

Ideal Gas For an ideal gas $P=\rho RT$ and $e=e(T)$ only. In that case, we have

$$h(T) = e + RT = h_\circ + \int_{T_\circ}^{T} c_v(T)\,\mathrm{d}T, \qquad s = s_\circ + \int_{T_\circ}^{T}\frac{c_P(T)}{T}\,\mathrm{d}T - R\ln(P/P_\circ) \tag{150}$$

and you can show that Γ is given by:

$$\Gamma^{ig} = \frac{\gamma+1}{2} + \frac{\gamma-1}{2}\frac{T}{\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}T} \tag{151}$$

Perfect or Constant-γ Gas Perfect gas results for isentropic flow can be derived from the equation of state

$$P=\rho RT \qquad h=c_pT \qquad c_p=\frac{\gamma R}{\gamma-1} \tag{152}$$

the value of Γ for a perfect gas,

$$\Gamma^{pg} = \frac{\gamma+1}{2} \tag{153}$$

the energy integral,

$$T_t = T\left(1+\frac{\gamma-1}{2}M^2\right) \tag{154}$$

and the expression for entropy

$$s - s_o = c_p\ln\frac{T}{T_o} - R\ln P/P_o \tag{155}$$

or

$$s - s_o = c_v \ln \frac{T}{T_o} - R \ln \rho/\rho_o$$

$$\frac{T_t}{T} = 1 + \frac{\gamma - 1}{2} M^2 \tag{156}$$

$$\frac{P_t}{P} = \left(\frac{T_t}{T}\right)^{\frac{\gamma}{\gamma-1}} \tag{157}$$

$$\frac{\rho_t}{\rho} = \left(\frac{T_t}{T}\right)^{\frac{1}{\gamma-1}} \tag{158}$$

Mach Number–Area Relationship

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{2}{\gamma+1}\left(1 + \frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} \tag{159}$$

Choked flow mass flux

$$\dot{M} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} c_t \rho_t A^* \tag{160}$$

or

$$\dot{M} = \sqrt{\gamma}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \frac{P_t}{\sqrt{RT_t}} A^*$$

Velocity-Mach number relationship

$$u = c_t \frac{M}{\sqrt{1 + \frac{\gamma-1}{2}M^2}} \tag{161}$$

Alternative reference speeds

$$c_t = c^* \sqrt{\frac{\gamma+1}{2}} \qquad u_{max} = c^* \sqrt{\frac{\gamma+1}{\gamma-1}} \tag{162}$$

3.3 Heat and Friction

Constant-area, steady flow with friction F and heat addition Q

$$\rho u = \dot{m} = \text{constant} \tag{163}$$

$$\rho u \mathrm{d}u + \mathrm{d}P = -F \mathrm{d}x \tag{164}$$

$$\mathrm{d}h + u \mathrm{d}u = Q \mathrm{d}x \tag{165}$$

$$\mathrm{d}s = \frac{1}{T}\left(Q + \frac{F}{\rho}\right) \mathrm{d}x \tag{166}$$

F is the frictional stress per unit length of the duct. In terms of the *Fanning friction factor* f

$$F = \frac{2}{D} f \rho u^2 \tag{167}$$

where D is the hydraulic diameter of the duct $D = 4\times$area/perimeter. Note that the conventional D'Arcy or Moody friction factor $\lambda = 4\,f$.

Q is the energy addition as heat per unit mass and unit length of the duct. If the heat flux into the fluid is $\dot{q}$, then we have

$$Q = \frac{\dot{q}}{\rho u}\frac{4}{D} \tag{168}$$

3.3.1 Fanno Flow

Constant-area, adiabatic, steady flow with friction only:

$$\rho u = \dot{m} = \text{constant} \tag{169}$$

$$\rho u \mathrm{d}u + \mathrm{d}P = -F \mathrm{d}x \tag{170}$$

$$h + \frac{u^2}{2} = h_t = \text{constant} \tag{171}$$

$$\tag{172}$$

Change in entropy with volume along *Fanno* line, $h + 1/2\dot{m}^2 v^2 {=} h_t$

$$T\left.\frac{\mathrm{d}s}{\mathrm{d}v}\right)_{Fanno} = \frac{c^2 - u^2}{v(1+G)} \tag{173}$$

3.3.2 Rayleigh Flow

Constant-area, steady flow with heat transfer only:

$$\rho u = \dot{m} = \text{constant} \tag{174}$$

$$P + \rho u^2 = I \tag{175}$$

$$\mathrm{d}h + u\mathrm{d}u = Q\mathrm{d}x \tag{176}$$

$$\tag{177}$$

Change in entropy with volume along *Rayleigh* line, $P + \dot{m}^2 v = I$

$$T\left.\frac{\mathrm{d}s}{\mathrm{d}v}\right)_{Rayleigh} = \frac{c^2 - u^2}{vG} \tag{178}$$

3.4 Shock Jump Conditions

The basic jump conditions,

$$
\begin{aligned}
\rho_1 w_1 &= \rho_2 w_2 && (179)\\
P_1 + \rho_1 w_1^2 &= P_2 + \rho_2 w_2^2 && (180)\\
h_1 + \frac{w_1^2}{2} &= h_2 + \frac{w_2^2}{2} && (181)\\
s_2 &\geq s_1 && (182)
\end{aligned}
$$

or defining $[f] \equiv f_2$ - f_1

$$
\begin{aligned}
[\rho w] &= 0 && (183)\\
\left[P + \rho w^2\right] &= 0 && (184)\\
\left[h + \frac{w^2}{2}\right] &= 0 && (185)\\
[s] &\geq 0 && (186)
\end{aligned}
$$

The Rayleigh line:

$$
\frac{P_2 - P_1}{v_2 - v_1} = -(\rho_1 w_1)^2 = -(\rho_2 w_2)^2 \qquad (187)
$$

or

$$
\frac{[P]}{[v]} = -(\rho w)^2 \qquad (188)
$$

Rankine-Hugoniot relation:

$$
h_2 - h_1 = (P_2 - P_1)(v_2 + v_1)/2 \qquad \text{or} \qquad e_2 - e_1 = (P_2 + P_1)(v_1 - v_2)/2 \qquad (189)
$$

Velocity-Pv relation

$$
[w]^2 = -[P][v] \qquad \text{or} \qquad w_2 - w_1 = -\sqrt{-(P_2 - P_1)(v_2 - v_1)} \qquad (190)
$$

Alternate relations useful for numerical solution

$$
\begin{aligned}
P_2 &= P_1 + \rho_1 w_1^2 \left(1 - \frac{\rho_1}{\rho_2}\right) && (191)\\
h_2 &= h_1 + \frac{1}{2} w_1^2 \left[1 - \left(\frac{\rho_1}{\rho_2}\right)^2\right] && (192)
\end{aligned}
$$

3.4.1 Lab frame (moving shock) versions

Shock velocity

$$w_1 = U_s \tag{193}$$

Particle (fluid) velocity in laboratory frame

$$w_2 = U_s - u_p \tag{194}$$

Jump conditions

$$\begin{aligned} \rho_2 \left(U_s - u_p\right) &= \rho_1 U_s & (195) \\ P_2 &= P_1 + \rho_1 U_s u_p & (196) \\ h_2 &= h_1 + u_p \left(U_s - u_p/2\right) & (197) \end{aligned}$$

Kinetic energy:

$$\frac{u_p^2}{2} = \frac{1}{2}(P_2 - P_1)(v_1 - v_2)$$

3.5 Perfect Gas Results

$$\begin{aligned} \frac{[P]}{P_1} &= \frac{2\gamma}{\gamma+1}\left(M_1^2 - 1\right) & (198) \\ \frac{[w]}{c_1} &= -\frac{2}{\gamma+1}\left(M_1 - \frac{1}{M_1}\right) & (199) \\ \frac{[v]}{v_1} &= -\frac{2}{\gamma+1}\left(1 - \frac{1}{M_1^2}\right) & (200) \\ \frac{[s]}{R} &= -\ln\frac{P_{t2}}{P_{t1}} & (201) \\ \frac{P_{t2}}{P_{t1}} &= \frac{1}{\left(\frac{2\gamma}{\gamma+1}M_1^2 - \frac{\gamma-1}{\gamma+1}\right)^{\frac{1}{\gamma-1}}}\left(\frac{\frac{\gamma+1}{2}M_1^2}{1 + \frac{\gamma-1}{2}M_1^2}\right)^{\frac{\gamma}{\gamma-1}} & (202) \end{aligned}$$

Shock adiabat or Hugoniot:

$$\frac{P_2}{P_1} = \frac{\frac{\gamma+1}{\gamma-1} - \frac{v_2}{v_1}}{\frac{\gamma+1}{\gamma-1}\frac{v_2}{v_1} - 1} \tag{203}$$

Some alternatives

$$\frac{P_2}{P_1} = 1 + \frac{2\gamma}{\gamma+1}\left(M_1^2 - 1\right) \tag{204}$$

$$= \frac{2\gamma}{\gamma+1}M_1^2 - \frac{\gamma-1}{\gamma+1} \tag{205}$$

$$\frac{\rho_2}{\rho_1} = \frac{\gamma+1}{\gamma - 1 + 2/M_1^2} \tag{206}$$

$$M_2^2 = \frac{M_1^2 + \dfrac{2}{\gamma-1}}{\dfrac{2\gamma}{\gamma-1}M_1^2 - 1} \tag{207}$$

Prandtl's relation

$$w_1 w_2 = c^{*2} \tag{208}$$

where c^* is the sound speed at a sonic point obtained in a fictitious isentropic process in the upstream flow.

$$c^* = \sqrt{2\frac{\gamma-1}{\gamma+1}h_t}, \qquad h_t = h + \frac{w^2}{2} \tag{209}$$

3.6 Reflected Shock Waves

Reflected shock velocity U_R in terms of the velocity u_2 and density ρ_2 behind the incident shock or detonation wave, and the density ρ_3 behind the reflected shock.

$$U_R = \frac{u_2}{\dfrac{\rho_3}{\rho_2} - 1} \tag{210}$$

Pressure P_3 behind reflected shock:

$$P_3 = P_2 + \frac{\rho_3 u_2^2}{\dfrac{\rho_3}{\rho_2} - 1} \tag{211}$$

Enthalpy h_3 behind reflected shock:

$$h_3 = h_2 + \frac{u_2^2}{2}\frac{\dfrac{\rho_3}{\rho_2} + 1}{\dfrac{\rho_3}{\rho_2} - 1} \tag{212}$$

Perfect gas result for incident shock waves:

$$\frac{P_3}{P_2} = \frac{(3\gamma-1)\dfrac{P_2}{P_1} - (\gamma-1)}{(\gamma-1)\dfrac{P_2}{P_1} + (\gamma+1)} \tag{213}$$

3.7 Detonation Waves

Jump conditions:

$$\rho_1 w_1 = \rho_2 w_2 \tag{214}$$
$$P_1 + \rho_1 w_1^2 = P_2 + \rho_2 w_2^2 \tag{215}$$
$$h_1 + \frac{w_1^2}{2} = h_2 + \frac{w_2^2}{2} \tag{216}$$
$$s_2 \geq s_1 \tag{217}$$

3.8 Perfect-Gas, 2-γ Model

Perfect gas with energy release q, different values of γ and R in reactants and products.

$$h_1 = c_{p1} T \tag{218}$$
$$h_2 = c_{p2} T - q \tag{219}$$
$$P_1 = \rho_1 R_1 T_1 \tag{220}$$
$$P_2 = \rho_2 R_2 T_2 \tag{221}$$
$$c_{p1} = \frac{\gamma_1 R_1}{\gamma_1 - 1} \tag{222}$$
$$c_{p2} = \frac{\gamma_2 R_2}{\gamma_2 - 1} \tag{223}$$
$$\tag{224}$$

Substitute into the jump conditions to yield:

$$\frac{P_2}{P_1} = \frac{1 + \gamma_1 M_1^2}{1 + \gamma_2 M_2^2} \tag{225}$$

$$\frac{v_2}{v_1} = \frac{\gamma_2 M_2^2}{\gamma_1 M_1^2} \frac{1 + \gamma_1 M_1^2}{1 + \gamma_2 M_2^2} \tag{226}$$

$$\frac{T_2}{T_1} = \frac{\gamma_1 R_1}{\gamma_2 R_2} \frac{\frac{1}{\gamma_1 - 1} + \frac{1}{2} M_1^2 + \frac{q}{c_1^2}}{\frac{1}{\gamma_2 - 1} + \frac{1}{2} M_2^2} \tag{227}$$

Chapman-Jouguet Conditions Isentrope, Hugoniot and Rayleigh lines are all tangent at the CJ point

$$\frac{P_{CJ} - P_1}{v_{CJ} - V_1} = \left.\frac{\partial P}{\partial v}\right)_{Hugoniot} = \left.\frac{\partial P}{\partial v}\right)_s \tag{228}$$

which implies that the product velocity is *sonic relative to the wave*

$$w_{2,CJ} = c_2 \tag{229}$$

Entropy variation along adiabat

$$ds = \frac{1}{2T}(v_1 - v)^2 \, d\dot{m}^2 \tag{230}$$

Jouguet's Rule

$$\frac{w^2 - c^2}{v^2} = \left[1 - \frac{G}{2v}(v_1 - v)\right]\left[\left.\frac{\partial P}{\partial v}\right)_{Hug} - \frac{\Delta P}{\Delta v}\right] \tag{231}$$

where G is the Grúniesen coefficient.

The flow downstream of a detonation is subsonic relative to the wave for points above the CJ state and supersonic for states below.

3.8.1 2-γ Solution

Mach Number for upper CJ (detonation) point

$$M_{CJ} = \sqrt{\mathcal{H} + \frac{(\gamma_1 + \gamma_2)(\gamma_2 - 1)}{2\gamma_1(\gamma_1 - 1)}} + \sqrt{\mathcal{H} + \frac{(\gamma_2 - \gamma_1)(\gamma_2 + 1)}{2\gamma_1(\gamma_1 - 1)}} \tag{232}$$

where the parameter $\mathcal{H}$ is the nondimensional energy release

$$\mathcal{H} = \frac{(\gamma_2 - 1)(\gamma_2 + 1)q}{2\gamma_1 R_1 T_1} \tag{233}$$

CJ pressure

$$\frac{P_{CJ}}{P_1} = \frac{\gamma_1 M_{CJ}^2 + 1}{\gamma_2 + 1} \tag{234}$$

CJ density

$$\frac{\rho_{CJ}}{\rho_1} = \frac{\gamma_1(\gamma_2 + 1)M_{CJ}^2}{\gamma_2(1 + \gamma_1 M_{CJ}^2)} \tag{235}$$

CJ temperature

$$\frac{T_{CJ}}{T_1} = \frac{P_{CJ}}{P_1}\frac{R_1\rho_1}{R_2\rho_{CJ}} \tag{236}$$

Strong detonation approximation $M_{CJ} \gg 1$

$$U_{CJ} \approx \sqrt{2(\gamma_2^2 - 1)q} \tag{237}$$

$$\rho_{CJ} \approx \frac{\gamma_2 + 1}{\gamma_2}\rho_1 \tag{238}$$

$$P_{CJ} \approx \frac{1}{\gamma_2 + 1}\rho_1 U_{CJ}^2 \tag{239}$$

$$\tag{240}$$

3.8.2 High-Explosives

For high-explosives, the same jump conditions apply but the ideal gas equation of state is no longer appropriate for the products. A simple way to deal with this problem is through the nondimensional slope γ_s of the *principal isentrope*, i.e., the isentrope passing through the CJ point:

$$\gamma_s \equiv -\frac{v}{P}\left.\frac{\partial P}{\partial v}\right)_s \tag{241}$$

Note that for a perfect gas, γ_s is identical to $\gamma = c_p/c_v$, the ratio of specific heats. In general, if the principal isentrope can be expressed as a power law:

$$Pv^k = \text{constant} \tag{242}$$

then $\gamma_s = k$. For high explosive products, $\gamma_s \approx 3$. From the definition of the CJ point, we have that the slope of the Rayleigh line and isentrope are equal at the CJ point:

$$\left.\frac{\partial P}{\partial v}\right)_s = \frac{P_{\mathrm{CJ}} - P_1}{v_{\mathrm{CJ}} - V_1} = -\frac{P_{\mathrm{CJ}}}{v_{\mathrm{CJ}}}\gamma_{s,\mathrm{CJ}} \tag{243}$$

From the mass conservation equation,

$$v_{\mathrm{CJ}} = v_1 \frac{\gamma_{s,\mathrm{CJ}}}{\gamma_{s,\mathrm{CJ}} + 1} \tag{244}$$

and from momentum conservation, with $P_{\mathrm{CJ}} \gg P_1$, we have

$$P_{\mathrm{CJ}} = \frac{\rho_1 U_{\mathrm{CJ}}^2}{\gamma_{s,\mathrm{CJ}} + 1} \tag{245}$$

3.9 Weak shock waves

Nondimensional pressure jump

$$\Pi = \frac{[P]}{\rho c^2} \tag{246}$$

A useful version of the jump conditions (exact):

$$\Pi = -M_1 \frac{[w]}{c_1} = -M_1^2 \frac{[v]}{v_1} \qquad \frac{[w]}{c_1} = M_1 \frac{[v]}{v_1} \tag{247}$$

Thermodynamic expansions:

$$\frac{[v]}{v_1} = -\Pi + \Gamma\Pi^2 + O(\Pi)^3 \tag{248}$$

$$\Pi = -\frac{[v]}{v_1} + \Gamma \left(\frac{[v]}{v_1}\right)^2 + O\left([v]\right)^3 \tag{249}$$

Linearized jump conditions:

$$-\frac{[w]}{c_1} = \Pi - \frac{\Gamma}{2}\Pi^2 + O(\Pi)^3 \tag{250}$$

$$M_1 = 1 - \frac{\Gamma}{2}\frac{[w]}{c_1} + O\left(\frac{[w]}{c_1}\right)^2 \tag{251}$$

$$M_1 = 1 + \frac{\Gamma}{2}\Pi + O(\Pi)^2 \tag{252}$$

$$M_2 = 1 - \frac{\Gamma}{2}\Pi + O(\Pi)^2 \tag{253}$$

$$\frac{[c]}{c_1} = (\Gamma - 1)\Pi + O(\Pi)^2 \tag{254}$$

$$M_1 - 1 \approx 1 - M_2 \tag{255}$$

Prandtl's relation

$$c^* \approx w_1 + \frac{1}{2}[w] \quad \text{or} \quad \approx w_2 - \frac{1}{2}[w] \tag{256}$$

Change in entropy for weak waves:

$$\frac{T[s]}{c_1^2} = \frac{1}{6}\Gamma\Pi^3 + \ldots \quad \text{or} = -\frac{1}{6}\Gamma\left(\frac{[v]}{v}\right)^3 + \ldots \tag{257}$$

3.10 Acoustics

Simple waves

$$\Delta P = c^2 \Delta \rho \tag{258}$$

$$\Delta P = \pm \rho c \Delta u \tag{259}$$

+ for right-moving waves, - for left-moving waves
Acoustic Potential ϕ

$$\begin{aligned} \mathbf{u} &= \boldsymbol{\nabla}\phi && (260)\\ P' &= -\rho_o \frac{\partial \phi}{\partial t} && (261)\\ \rho' &= -\frac{\rho_o}{c_o^2}\frac{\partial \phi}{\partial t} && (262) \end{aligned}$$

Potential Equation

$$\nabla^2 \phi - \frac{1}{c_o^2}\frac{\partial^2 \phi}{\partial t^2} = 0 \tag{263}$$

d'Alembert's solution for planar (1D) waves

$$\phi = f(x - c_o t) + g(x + c_o t) \tag{264}$$

Acoustic Impedance The specific *acoustic impedance* of a medium is defined as

$$z = \frac{P'}{|\mathbf{u}|} \tag{265}$$

For a planar wavefront in a homogeneous medium $z = \pm \rho c$, depending on the direction of propagation.

Transmission coefficients A plane wave in medium 1 is normally incident on an interface with medium 2. Incident (i) and transmitted wave (t)

$$\begin{aligned} u_t/u_i &= \frac{2z_1}{z_2 + z_1} && (266)\\ P'_t/P'_i &= \frac{2z_2}{z_2 + z_1} && (267) \end{aligned}$$

Harmonic waves (planar)

$$\phi = A \exp i(wt - kx) + B \exp i(wt + kx) \qquad c = \frac{\omega}{k} \qquad k = \frac{2\pi}{\lambda} \qquad \omega = \frac{2\pi}{T} = 2\pi f \tag{268}$$

Spherical waves

$$\phi = \frac{f(t - r/c)}{r} + \frac{g(t + r/c)}{r} \tag{269}$$

Spherical source strength Q, $[Q] = L^3T^{-1}$

$$Q(t) = \lim_{r\to 0} 4\pi r^2 u_r \tag{270}$$

potential function

$$\phi(r,t) = -\frac{Q(t - r/c)}{4\pi r} \tag{271}$$

Energy flux

$$\Phi = P'\mathbf{u} \tag{272}$$

Acoustic intensity for harmonic waves

$$I =< \Phi >= \frac{1}{T}\int_0^T \Phi\,\mathrm{d}t = \frac{P'^2_{rms}}{\rho c} \tag{273}$$

Decibel scale of acoustic intensity

$$dB = 10\log_{10}(I/I_{ref}) \qquad I_{ref} = 10^{-12} \quad \mathrm{W/m^2} \tag{274}$$

or

$$dB = 20\log_{10}(P'_{rms}/P'_{ref}) \qquad P'_{ref} = 2\times 10^{-10} \quad \mathrm{atm} \tag{275}$$

Cylindrical waves, q source strength per unit length $[q] = L^2T^{-1}$

$$\phi(r,t) = -\frac{1}{2\pi}\int_{-\infty}^{t-r/c} \frac{q(\eta)\,\mathrm{d}\eta}{\sqrt{(t-\eta)^2 - r^2/c^2}} \tag{276}$$

or

$$\phi(r,t) = -\frac{1}{2\pi}\int_0^{\infty} q(t - r/c\cosh\xi)\,\mathrm{d}\xi \tag{277}$$

3.11 Multipole Expansion

Potential from a distribution of volume sources, strength q per unit source volume

$$\phi(\mathbf{x},t) = -\frac{1}{4\pi}\int_{V_s} \frac{q(\mathbf{x}_s, t - R/c)}{R}\mathrm{d}V_s \qquad R = |\mathbf{x} - \mathbf{x}_s| \tag{278}$$

Harmonic source

$$q = f(\mathbf{x})\exp(-i\omega t)$$

Potential function

$$\phi(\mathbf{x},t) = -\frac{1}{4\pi}\int_{V_s} f(\mathbf{x}_s)\frac{\exp i(kR - \omega t)}{R}\mathrm{d}V_s \tag{279}$$

Compact source approximation:

1. source distribution is in bounded region around the origin $\mathbf{x}_s < a$, and small $a \ll r = |\mathbf{x}|$
2. source distribution is compact $ka << 1$ or $\lambda \gg a$, so that the phase factor $\exp ikR$ does not vary too much across the source

Multipole expansion:

$$\frac{\exp ikR}{R} = \sum_{n=0}^{\infty} \frac{(-\mathbf{x}_s \cdot \boldsymbol{\nabla}_{\mathbf{x}})^n}{n!} \left(\frac{\exp ikr}{r} \right) \tag{280}$$

Series expansion of potential

$$\phi = \phi^0 + \phi^1 + \phi^2 + \dots \tag{281}$$

Monopole term

$$\phi^0(\mathbf{x}, t) = -\frac{\exp i(kr - \omega t)}{4\pi r} \int_{V_s} f(\mathbf{x}_s) \mathrm{d}V_s \tag{282}$$

Dipole term

$$\phi^1(\mathbf{x}, t) = \frac{ik\mathbf{x} \cdot \mathbf{D}}{4\pi r^2} \left(1 + \frac{i}{kr} \right) \exp i(kr - \omega t) \tag{283}$$

Dipole moment vector $\mathbf{D}$

$$\mathbf{D} = \int_{V_s} \mathbf{x}_s f(\mathbf{x}_s) \mathrm{d}V_s \tag{284}$$

Quadrupole term

$$\phi^2(\mathbf{x}, t) = \frac{k^2}{4\pi r^3} \left(1 + \frac{3i}{kr} - \frac{3}{k^2 r^2} \right) \exp i(kr - \omega t) \sum_{i,j} x_i x_j Q_{ij} \tag{285}$$

Quadrupole moments Q_{ik}

$$Q_{ij} = \frac{1}{2} \int_{V_s} x_{s,i} x_{s,j} f(\mathbf{x}_s) \mathrm{d}V_s \tag{286}$$

3.12 Baffled (surface) source

Rayleigh's formula for the potential

$$\phi = -\frac{1}{2\pi} \int \frac{u_n(\mathbf{x}_s, t - R/c)}{R} \mathrm{d}A \tag{287}$$

Normal component of the source surface velocity

$$u_n = \mathbf{u} \cdot \hat{\mathbf{n}} \tag{288}$$

Harmonic source

$$u_n = f(\mathbf{x}) \exp(-iwt)$$

Fraunhofer conditions $|\mathbf{x}_s| \leq a$

$$\frac{a}{\lambda} \frac{a}{r} \ll 1$$

Approximate solution:

$$\phi = -\frac{\exp i(kr - wt)}{2\pi r} \int_{A_s} f(\mathbf{x}_s) \exp i\kappa \cdot \mathbf{x}_s \mathrm{d}A$$

3.13 1-D Unsteady Flow

The primitive variable version of the equations is:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0 \tag{289}$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}\mathbf{u}) = -\boldsymbol{\nabla} P \tag{290}$$

$$\frac{\partial}{\partial t}\rho \left(e + \frac{u^2}{2}\right) + \boldsymbol{\nabla} \cdot \left(\rho \mathbf{u}(h + \frac{u^2}{2})\right) = 0 \tag{291}$$

$$\frac{\partial s}{\partial t} + \boldsymbol{\nabla} \cdot (\mathbf{u} s) \geq 0 \tag{292}$$

$$\tag{293}$$

Alternative version

$$\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\boldsymbol{\nabla} \cdot \mathbf{u} \tag{294}$$

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\boldsymbol{\nabla} P \tag{295}$$

$$\rho \frac{\mathrm{D}}{\mathrm{D}t}\left(h + \frac{u^2}{2}\right) = \frac{\partial P}{\partial t} \tag{296}$$

$$\frac{\mathrm{D}s}{\mathrm{D}t} \geq 0 \tag{297}$$

The characteristic version of the equations for isentropic flow (s = constant) is:

$$\frac{\mathrm{d}}{\mathrm{d}t}(u \pm F) = 0 \quad \text{on} \quad C^{\pm} : \; \frac{\mathrm{d}x}{\mathrm{d}t} = u \pm c \tag{298}$$

This is equivalent to:

$$\frac{\partial}{\partial t}(u \pm F) + (u \pm c)\frac{\partial}{\partial x}(u \pm F) = 0 \tag{299}$$

Riemann invariants:

$$F = \int \frac{c}{\rho} \mathrm{d}\,\rho = \int \frac{\mathrm{d}\,P}{\rho c} = \int \frac{\mathrm{d}\,c}{\Gamma - 1} \tag{300}$$

Bending of characteristics:

$$\frac{\mathrm{d}}{\mathrm{d}P}(u + c) = \frac{\Gamma}{\rho c} \tag{301}$$

For an ideal gas:

$$F = \frac{2c}{\gamma - 1} \tag{302}$$

Pressure-velocity relationship for expansion waves moving to the right into state (1), final state (2) with velocity $u_2 < 0$.

$$\frac{P_2}{P_1} = \left(1 + \frac{\gamma - 1}{2}\frac{u_2}{c_1}\right)^{\frac{2\gamma}{\gamma-1}} \qquad \frac{-2c_1}{\gamma - 1} < u_2 \le 0 \tag{303}$$

Shock waves moving to the right into state (1), final state (2) with velocity $u_2 > 0$.

$$\frac{[P]}{P_1} = \frac{\gamma(\gamma+1)}{4}\left(\frac{u_2}{c_1}\right)^2 \left[1 + \sqrt{1 + \left(\frac{4}{\gamma+1}\frac{c_1}{u_2}\right)^2}\right) \qquad u_2 > 0 \tag{304}$$

Shock Tube Performance

$$\frac{P_4}{P_1} = \left[1 - \frac{c_1}{c_4}\frac{\gamma_4 - 1}{\gamma + 1}\left(M_s - \frac{1}{M_s}\right)\right]^{\frac{-2\gamma_4}{\gamma_4-1}} \left[1 + \frac{2\gamma_1}{\gamma_1 + 1}\left(M_s^2 - 1\right)\right] \tag{305}$$

Limiting shock Mach number for $P_4/P_1 \to \infty$

$$M_s \to \frac{c_4}{c_1}\frac{\gamma_1 + 1}{\gamma_4 - 1} \tag{306}$$

3.14 2-D Steady Flow

3.14.1 Oblique Shock Waves

Geometry:

$$\begin{aligned} w_1 &= u_1 \sin\beta && (307)\\ w_2 &= u_2 \sin(\beta - \theta) && (308)\\ v &= u_1 \cos\beta = u_2 \cos(\beta - \theta) && (309) \end{aligned}$$

$$\frac{\rho_2}{\rho_1} = \frac{w_1}{w_2} = \frac{\tan\beta}{\tan(\beta - \theta)} \tag{310}$$

Shock Polar

$$-\frac{[w]}{c_1} = \frac{M_1 \tan\theta}{\cos\beta(1 + \tan\beta\tan\theta)} \tag{311}$$

$$\frac{[P]}{\rho_1 c_1^2} = \frac{M_1^2 \tan\theta}{\cot\beta + \tan\theta} \tag{312}$$

Real fluid results

$$\begin{aligned} w_2 &= f(w_1) \qquad \text{normal shock jump conditions} && (313)\\ \beta &= \sin^{-1}(w_1/u_1) && (314)\\ \theta &= \beta - \tan^{-1}\left(\frac{w_2}{\sqrt{u_1^2 - w_1^2}}\right) && (315) \end{aligned}$$

Perfect gas result

$$\tan\theta = \frac{2\cot\beta\left(M_1^2\sin^2\beta - 1\right)}{(\gamma+1)M_1^2 - 2\left(M_1^2\sin^2\beta - 1\right)} \tag{316}$$

Mach angle

$$\mu = \sin^{-1}\frac{1}{M} \tag{317}$$

3.14.2 Weak Oblique Waves

Results are all for C^+ family of waves, take $\theta \rightarrow$ -θ for C^- family.

$$\beta = \mu - \frac{\Gamma_1}{2}\frac{1}{\sqrt{M_1^2-1}}\frac{[w]}{c_1} + O\left(\frac{[w]}{c_1}\right)^2 \tag{318}$$

$$\theta = -\frac{\sqrt{M_1^2-1}}{M_1^2}\frac{[w]}{c_1} + O\left(\frac{[w]}{c_1}\right)^2 \tag{319}$$

$$\frac{[P]}{\rho_1 c_1^2} = \frac{M_1^2}{\sqrt{M_1^2-1}}\theta + O(\theta)^2 \tag{320}$$

$$\frac{T_1[s]}{c_1^2} = \frac{\Gamma_1}{6}\frac{M_1^6}{(M_1^2-1)^{3/2}}\theta^3 + O(\theta)^4 \tag{321}$$

Perfect Gas Results

$$\frac{[P]}{P_1} = \frac{\gamma M_1^2}{\sqrt{M_1^2-1}}\theta + O(\theta)^2 \tag{322}$$

3.14.3 Prandtl-Meyer Expansion

$$\mathrm{d}\,\theta = -\sqrt{M_1^2-1}\frac{\mathrm{d}\,u_1}{u_1} \tag{323}$$

Function ω, $\mathrm{d}\,\theta$ = -$\mathrm{d}\,\omega$

$$\mathrm{d}\,\omega \equiv \frac{\sqrt{M^2-1}}{1+(\Gamma-1)M^2}\frac{\mathrm{d}\,M}{M} \tag{324}$$

Perfect gas result

$$\omega(M) = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left(\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)}\right) - \tan^{-1}\sqrt{M^2-1} \tag{325}$$

Maximum turning angle

$$\omega_{max} = \frac{\pi}{2}\left(\sqrt{\frac{\gamma+1}{\gamma-1}} - 1\right) \tag{326}$$

3.14.4 Inviscid Flow

Crocco-Vaszonyi Relation

$$\frac{\partial \mathbf{u}}{\partial t} + (\boldsymbol{\nabla}\times\mathbf{u})\times\mathbf{u} = T\boldsymbol{\nabla}S - \boldsymbol{\nabla}(h + \frac{u^2}{2}) \tag{327}$$

3.14.5 Potential Flow

Steady, homoentropic, homoenthalpic, inviscid:

$$\begin{aligned} \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) &= 0 && (328) \\ \boldsymbol{\nabla} \times \mathbf{u} &= 0 && (329) \\ h + \frac{u^2}{2} &= constant && (330) \end{aligned}$$

or with $\mathbf{u} = \boldsymbol{\nabla}\phi = (\phi_x, \phi_y)$

$$(\phi_x^2 - c^2)\phi_{xx} + (\phi_y^2 - c^2)\phi_{yy} + 2\phi_x\phi_y\phi_{xy} = 0 \quad (331)$$

Linearized potential flow:

$$\begin{aligned} u &= U_\infty + \phi'_x && (332) \\ v &= \phi'_y && (333) \\ 0 &= \left(M_\infty^2 - 1\right)\phi'_{xx} - \phi'_{yy} && (334) \end{aligned}$$

Wave equation solution

$$\lambda = \sqrt{M_\infty^2 - 1} \qquad \phi' = f(x - \lambda y) + g(x + \lambda y) \quad (335)$$

Boundary conditions for slender 2-D (Cartesian) bodies $y(x)$

$$f'(\xi) = -\frac{U_\infty}{\lambda}\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right)_\xi \qquad y \geq 0 \quad (336)$$

Prandtl-Glauert Scaling for subsonic flows

$$\phi(x, y) = \phi^{inc}(x, \sqrt{1 - M_\infty^2}y) \qquad \nabla^2\phi^{inc} = 0 \quad (337)$$

Prandtl-Glauert Rule

$$C_p = \frac{C_p^{inc}}{\sqrt{1 - M_\infty^2}} \quad (338)$$

3.14.6 Natural Coordinates

$$\begin{aligned} \frac{\partial}{\partial x} &= \cos\theta\frac{\partial}{\partial s} - \sin\theta\frac{\partial}{\partial n} && (339) \\ \frac{\partial}{\partial y} &= \sin\theta\frac{\partial}{\partial s} + \cos\theta\frac{\partial}{\partial n} && (340) \\ u &= U\cos\theta && (341) \\ v &= U\sin\theta && (342) \end{aligned}$$

The transformed equations of motion are:

$$\frac{\partial \rho U}{\partial s} + \rho U \frac{\partial \theta}{\partial n} = 0 \tag{343}$$

$$\rho U \frac{\partial U}{\partial s} + \frac{\partial P}{\partial s} = 0 \tag{344}$$

$$\rho U^2 \frac{\partial \theta}{\partial s} + \frac{\partial P}{\partial n} = 0 \tag{345}$$

$$\omega_z = U \frac{\partial \theta}{\partial s} - \frac{\partial U}{\partial n} = 0 \tag{346}$$

Curvature of stream lines, R = radius of curvature

$$\frac{\partial \theta}{\partial s} = \frac{1}{R} \tag{347}$$

Vorticity production

$$\omega_z = -\frac{1}{U \rho_o} \frac{\partial P_o}{\partial n} + \frac{(T - T_o)}{U} \frac{\partial S}{\partial n} \tag{348}$$

Elimination of pressure $dP = c^2 d\rho$

$$(M^2 - 1) \frac{\partial U}{\partial s} - U \frac{\partial \theta}{\partial n} = 0 \tag{349}$$

$$\frac{\partial U}{\partial n} - U \frac{\partial \theta}{\partial s} = 0 \tag{350}$$

3.14.7 Method of Characteristics

$$\frac{\partial}{\partial s}(\omega - \theta) + \frac{1}{\sqrt{M^2 - 1}} \frac{\partial}{\partial n}(\omega - \theta) = 0 \tag{351}$$

$$\frac{\partial}{\partial s}(\omega + \theta) - \frac{1}{\sqrt{M^2 - 1}} \frac{\partial}{\partial n}(\omega + \theta) = 0 \tag{352}$$

$$\tag{353}$$

Characteristic directions

$$C^{\pm} \qquad \frac{\mathrm{d}n}{\mathrm{d}s} = \pm \frac{1}{\sqrt{M^2 - 1}} = \pm \tan \mu \tag{354}$$

Invariants

$$J^{\pm} = \theta \mp \omega = \text{constant on} \quad C^{\pm} \tag{355}$$

4 Incompressible, Inviscid Flow

4.1 Velocity Field Decomposition

Split the velocity field into two parts: irrotational $\mathbf{u}_e$, and rotational (vortical) $\mathbf{u}_v$.

$$\mathbf{u} = \mathbf{u}_e + \mathbf{u}_v \tag{356}$$

Irrotational Flow Define the irrotational portion of the flow by the following two conditions:

$$\boldsymbol{\nabla} \times \mathbf{u}_e = 0 \tag{357}$$

$$\boldsymbol{\nabla} \cdot \mathbf{u}_e = e(\mathbf{x}, t) \qquad \text{volume source distribution} \tag{358}$$

This is satisfied by deriving $\mathbf{u}_e$ from a velocity potential ϕ

$$\mathbf{u}_e = \boldsymbol{\nabla}\phi \tag{359}$$

$$\nabla^2\phi = e(\mathbf{x}, t) \tag{360}$$

Rotational Flow Define the rotational part of the flow by:

$$\boldsymbol{\nabla} \cdot \mathbf{u}_v = 0 \tag{361}$$

$$\boldsymbol{\nabla} \times \mathbf{u}_v = \boldsymbol{\omega}(\mathbf{x}, t) \qquad \text{vorticity source distribution} \tag{362}$$

This is satisfied by deriving $\mathbf{u}_v$ from a vector potential $\mathbf{B}$

$$\mathbf{u}_v = \boldsymbol{\nabla} \times \mathbf{B} \tag{363}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \qquad \text{choice of gauge} \tag{364}$$

$$\nabla^2\mathbf{B} = -\boldsymbol{\omega}(\mathbf{x}, t) \tag{365}$$

4.2 Solutions of Laplace's Equation

The equation $\nabla^2\phi = e$ is known as *Laplaces* equation and can be solved by the technique of *Green's* functions:

$$\phi(\mathbf{x}, t) = \int_{\Omega_{\boldsymbol{\xi}}} G(\mathbf{x}|\boldsymbol{\xi}) e(\boldsymbol{\xi}, t) dV_{\boldsymbol{\xi}} \tag{366}$$

For a infinite domain, Green's function is the solution to

$$\nabla^2 G = \delta(\mathbf{x} - \boldsymbol{\xi}) \tag{367}$$

$$G = -\frac{1}{4\pi}\frac{1}{|\mathbf{x} - \boldsymbol{\xi}|} = -\frac{1}{4\pi r} \tag{368}$$

$$r = |\mathbf{r}| \qquad \mathbf{r} = \mathbf{x} - \boldsymbol{\xi} \tag{369}$$

This leads to the following solutions for the potentials

$$\phi(\mathbf{x},t) = -\frac{1}{4\pi}\int_{\Omega_{\boldsymbol{\xi}}} \frac{e(\boldsymbol{\xi},t)}{r}\mathrm{d}V_{\boldsymbol{\xi}} \tag{370}$$

$$\mathbf{B}(\mathbf{x},t) = \frac{1}{4\pi}\int_{\Omega_{\boldsymbol{\xi}}} \frac{\boldsymbol{\omega}(\boldsymbol{\xi},t)}{r}\mathrm{d}V_{\boldsymbol{\xi}} \tag{371}$$

The velocity fields are

$$\mathbf{u}_e(\mathbf{x},t) = \frac{1}{4\pi}\int_{\Omega_{\boldsymbol{\xi}}} \frac{\mathbf{r}e(\boldsymbol{\xi},t)}{r^3}\mathrm{d}V_{\boldsymbol{\xi}} \tag{372}$$

$$\mathbf{u}_v(\mathbf{x},t) = -\frac{1}{4\pi}\int_{\Omega_{\boldsymbol{\xi}}} \frac{\mathbf{r}\times\boldsymbol{\omega}(\boldsymbol{\xi},t)}{r^3}\mathrm{d}V_{\boldsymbol{\xi}} \tag{373}$$

If the domain is finite or there are surfaces (stationary or moving bodies, free surfaces, boundaries), then an additional component of velocity, $\mathbf{u}'$, must be added to insure that the boundary conditions (described subsequently) are satisfied. This additional component will be a source-free, $\boldsymbol{\nabla}\cdot\mathbf{u}' = 0$, irrotational $\boldsymbol{\nabla}\times\mathbf{u}' = 0$ field. The general solution for the velocity field will then be

$$\mathbf{u} = \mathbf{u}_e + \mathbf{u}_v + \mathbf{u}' \tag{374}$$

4.3 Boundary Conditions

Solid Boundaries In general, at an impermeable boundary $\partial\Omega$, there is no relative motion between the fluid and boundary in the local direction $\hat{\mathbf{n}}$ normal to the boundary surface.

$$\mathbf{u}\cdot\hat{\mathbf{n}} = \mathbf{u}_{\partial\Omega}\cdot\hat{\mathbf{n}} \quad \text{on the surface } \partial\Omega \tag{375}$$

In particular, if the surface is stationary, the normal component of velocity must vanish on the surface

$$\mathbf{u}\cdot\hat{\mathbf{n}} = 0 \quad \text{on a stationary surface } \partial\Omega \tag{376}$$

For an *ideal* or *inviscid* fluid, there is no restriction on the velocity tangential to the boundary, *slip* boundary conditions.

$$\mathbf{u}\cdot\hat{\mathbf{t}} \quad \text{arbitrary on the surface } \partial\Omega \tag{377}$$

For a *real* or *viscous* fluid, the tangential component is zero, since the relative velocity between fluid and surface must vanish, the *no-slip* condition.

$$\mathbf{u} = 0 \quad \text{on the surface } \partial\Omega \tag{378}$$

Fluid Boundaries At an internal or free surface of an ideal fluid, the normal components of the velocity have to be equal on each side of the surface

$$\mathbf{u}_1 \cdot \hat{\mathbf{n}} = \mathbf{u}_2 \cdot \hat{\mathbf{n}} = \mathbf{u}_{\partial\Omega} \cdot \hat{\mathbf{n}} \tag{379}$$

and the interface has to be in mechanical equilibrium (in the absence of surface forces such as interfacial tension)

$$P_1 = P_2 \tag{380}$$

4.4 Streamfunction

The vector potential in flows that are two dimensional or have certain symmetries can be simplified to one component that can be represented as a scalar function known as the *streamfunction* ψ. The exact form of the streamfunction depends on the nature of the symmetry and related system of coordinates.

4.4.1 2-D Cartesian Flows

Compressible In a steady two-dimensional compressible flow:

$$\nabla \cdot \rho\mathbf{u} = 0 \qquad \mathbf{u} = (u, v) \qquad \mathbf{x} = (x, y) \qquad \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \tag{381}$$

The streamfunction is:

$$u = \frac{1}{\rho}\frac{\partial \psi}{\partial y} \qquad v = -\frac{1}{\rho}\frac{\partial \psi}{\partial x} \tag{382}$$

Incompressible The density ρ is a constant

$$\nabla \cdot \mathbf{u} = 0 \qquad \mathbf{u} = (u, v) \qquad \mathbf{x} = (x, y) \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{383}$$

The streamfunction defined by

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x} \tag{384}$$

will identically satisfy the continuity equation as long as

$$\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0 \tag{385}$$

which is always true as long as the function $\psi(x, y)$ has continuous 2^{nd} derivatives.

Stream lines (or surfaces in 3-D flows) are defined by $\psi =$ constant. The normal to the stream surface is

$$\hat{\mathbf{n}}_\psi = \frac{\nabla \psi}{|\nabla \psi|} \tag{386}$$

Integration of the differential of the stream function along a path $\mathcal{L}$ connecting points $\mathbf{x}_1$ and $\mathbf{x}_2$ in the plane can be interpreted as volume flux across the path

$$\mathrm{d}\psi = \mathbf{u}\cdot\hat{\mathbf{n}}_{\mathcal{L}}\mathrm{d}l = -v\,\mathrm{d}x + u\,\mathrm{d}y \tag{387}$$

$$\int_{\mathcal{L}} \mathrm{d}\psi = \psi_2 - \psi_1 = \int_{\mathcal{L}} \mathbf{u}\cdot\hat{\mathbf{n}}_{\mathcal{L}}\mathrm{d}l = \text{volume flux across} \quad \mathcal{L} \tag{388}$$

where $\psi_1 = \psi(\mathbf{x}_1)$ and $\psi_2 = \psi(\mathbf{x}_2)$. For compressible flows, the difference in the streamfunction can be interpreted as the mass flux rather than the volume flux.

For this flow, the streamfunction is exactly the nonzero component of the vector potential

$$\mathbf{B} = (B_x, B_y, B_z) = (0, 0, \psi) \qquad \mathbf{u} = \boldsymbol{\nabla}\times\mathbf{B} = \hat{\mathbf{x}}\frac{\partial\psi}{\partial y} - \hat{\mathbf{y}}\frac{\partial\psi}{\partial x} \tag{389}$$

and the equation that the streamfunction has to satisfy will be

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = -\omega_z \tag{390}$$

where the z-component of vorticity is

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{391}$$

A special case of this is irrotational flow with $\omega_z = 0$.

4.4.2 Cylindrical Polar Coordinates

In cylindrical polar coordinates (r, θ, z) with $\mathbf{u} = (u_r, u_\theta, u_z)$

$$x = r\cos\theta \tag{392}$$

$$y = r\sin\theta \tag{393}$$

$$z = z \tag{394}$$

$$u = u_r\cos\theta - u_\theta\sin\theta \tag{395}$$

$$v = u_r\sin\theta + u_\theta\cos\theta \tag{396}$$

$$w = u_z \tag{397}$$

The continuity equation is

$$\boldsymbol{\nabla}\cdot\mathbf{u} = 0 = \frac{1}{r}\frac{\partial r u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_z}{\partial z} \tag{398}$$

Translational Symmetry in z The results given above for 2-D incompressible flow have translational symmetry in z such that $\partial/\partial z = 0$. These can be rewritten in terms of the streamfunction $\psi(r, \theta)$ where

$$\mathbf{B} = (0, 0, \psi) \tag{399}$$

The velocity components are

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \tag{400}$$

$$u_\theta = -\frac{\partial \psi}{\partial r} \tag{401}$$

The only nonzero component of vorticity is

$$\omega_z = \frac{1}{r}\frac{\partial r u_\theta}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta} \tag{402}$$

and the stream function satisfies

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial \psi}{\partial \theta}\right) = -\omega_z \tag{403}$$

Rotational Symmetry in θ If the flow has rotational symmetry in θ, such that $\partial/\partial\theta = 0$, then the stream function can be defined as

$$\mathbf{B} = \left(0, \frac{\psi}{r}, 0\right) \tag{404}$$

and the velocity components are:

$$u_r = -\frac{1}{r}\frac{\partial \psi}{\partial z} \tag{405}$$

$$u_z = \frac{1}{r}\frac{\partial \psi}{\partial r} \tag{406}$$

The only nonzero vorticity component is

$$\omega_\theta = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \tag{407}$$

The stream function satisfies

$$\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial \psi}{\partial z}\right) + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) = -\omega_\theta \tag{408}$$

4.4.3 Spherical Polar Coordinates

This coordinate system (r, ϕ, θ) results in the continuity equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r\sin\phi}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r\sin\phi}\frac{\partial}{\partial \phi}\left(u_\phi \sin\phi\right) = 0 \tag{409}$$

Note that the r coordinate in this system is defined differently than in the cylindrical polar system discussed previously. If we denote by r' the radial distance from the z-axis in the cylindrical polar coordinates, then $r' = r\sin\phi$. With symmetry in the θ direction $\partial/\partial\theta$, the following *Stokes'* stream function can be defined

$$\mathbf{B} = \left(0, 0, \frac{\psi}{r\sin\phi}\right) \tag{410}$$

Note that this stream function is identical to that used in the previous discussion of the case of rotational symmetry in θ for the cylindrical polar coordinate system if we account for the reordering of the vector components and the differences in the definitions of the radial coordinates.

The velocity components are:

$$u_r = \frac{1}{r^2\sin\phi}\frac{\partial\psi}{\partial\phi} \tag{411}$$

$$u_\phi = -\frac{1}{r\sin\phi}\frac{\partial\psi}{\partial r} \tag{412}$$

The only non-zero vorticity component is:

$$\omega_\theta = \frac{1}{r}\frac{\partial r u_\phi}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial\phi} \tag{413}$$

The stream function satisfies

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{\sin\phi}\frac{\partial\psi}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial\phi}\left(\frac{1}{r^2\sin\phi}\frac{\partial\psi}{\partial\phi}\right) = -\omega_\theta \tag{414}$$

4.5 Simple Flows

The simplest flows are source-free and irrotational, which can be derived by a potential that satisfies the *Laplace* equation, a special case of $\mathbf{u}_e$

$$\nabla^2\phi = 0 \qquad \boldsymbol{\nabla}\cdot\mathbf{u} = 0 \tag{415}$$

In the case of flows, that contain sources and sinks or other singularities, this equation holds everywhere except at those singular points.

Uniform Flow The simplest solution is a uniform flow $\mathbf{U}$:

$$\phi = \mathbf{U}\cdot\mathbf{x} \qquad \mathbf{u} = \mathbf{U} = \text{constant} \tag{416}$$

In 2-D cartesian coordinates with $\mathbf{U} = U\hat{\mathbf{x}}$, the streamfunction is

$$\psi = Uy \tag{417}$$

In spherical polar coordinates, Stokes streamfunction is

$$\psi = \frac{Ur^2}{2}\sin^2\phi \qquad \mathbf{U} = U\hat{\mathbf{z}} \tag{418}$$

Source Distributions Single source of strength $Q(t)$ located at point $\boldsymbol{\xi}_1$. The meaning of Q is the *volume of fluid per unit time* introduced or removed at point $\boldsymbol{\xi}_1$.

$$\lim_{r_1 \to 0} 4\pi r_1^2 \mathbf{u} \cdot \hat{\mathbf{r}}_1 = Q(t) \qquad \mathbf{r}_1 = \mathbf{x} - \boldsymbol{\xi}_1 \qquad e = Q(t)\delta(\mathbf{x} - \boldsymbol{\xi}_1) \tag{419}$$

which leads to the solution:

$$\phi = -\frac{Q(t)}{4\pi r_1} \qquad \mathbf{u} = \frac{\mathbf{r}_1 Q(t)}{4\pi r_1^3} = \frac{\hat{\mathbf{r}}_1 Q(t)}{4\pi r_1^2} \tag{420}$$

For multiple sources, add the individual solutions

$$\mathbf{u} = -\frac{1}{4\pi} \sum_{i=1}^{k} \frac{\mathbf{r}_i Q_i}{r_i^3} \tag{421}$$

In spherical polar cordinates, Stokes' stream function for a single source of strength Q at the origin is

$$\psi = -\frac{Q}{4\pi} \cos\phi \tag{422}$$

For a **2-D flow**, the source strength q is the volume flux per unit length or area per unit time since the source can be thought of as a *line* source.

$$\mathbf{u} = u_r \hat{\mathbf{r}} \qquad u_r = \frac{q}{2\pi r} \qquad \phi = \frac{q}{2\pi} \ln r \qquad \psi = \frac{q}{2\pi}\theta \tag{423}$$

Dipole Consider a source-sink pair of equal strength Q located a distance δ apart. The limiting process

$$\delta \to 0 \qquad Q \to \infty \qquad \delta Q \to \mu \tag{424}$$

defines a dipole of strength μ. If the direction from the sink to the source is $\hat{\mathbf{d}}$, then the dipole moment vector can be defined as

$$\mathbf{d} = \mu \hat{\mathbf{d}} \tag{425}$$

The dipole potential for spherical (3-D) sources is

$$\phi = -\frac{\mathbf{d} \cdot \mathbf{r}}{4\pi r^3} \tag{426}$$

and the resulting velocity field is

$$\mathbf{u} = \frac{1}{4\pi} \left[\frac{3\mathbf{d} \cdot \mathbf{r}}{r^5} \mathbf{r} - \frac{\mathbf{d}}{r^3} \right] \tag{427}$$

If the dipole is aligned with the z-axis, Stokes' stream function is

$$\psi = \frac{\mu \sin^2 \phi}{4\pi r} \tag{428}$$

and the velocity components are

$$u_r = \frac{\mu \cos\phi}{2\pi r^3} \quad (429)$$

$$u_\phi = \frac{\mu \sin\phi}{4\pi r^3} \quad (430)$$

The dipole potential for 2-D source-sink pairs is

$$\phi = -\frac{\mu}{2\pi}\frac{\cos\theta}{r} \quad (431)$$

and the stream function is

$$\psi = \frac{\mu}{2\pi}\frac{\sin\theta}{r} \quad (432)$$

The velocity components are

$$u_r = \frac{\mu}{2\pi}\frac{\cos\theta}{r^2} \quad (433)$$

$$u_\theta = \frac{\mu}{2\pi}\frac{\sin\theta}{r^2} \quad (434)$$

Combinations More complex flows can be built up by superposition of the flows discussed above. In particular, flows over bodies can be found as follows:

half-body:	source + uniform flow
sphere:	dipole (3-D) + uniform flow
cylinder:	dipole (2-D) + uniform flow
closed-body:	sources & sinks + uniform flow

4.6 Vorticity

Vorticity fields are divergence free In general, we have $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) \equiv 0$ so that the vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\mathbf{u}$, satisfies

$$\boldsymbol{\nabla}\cdot\boldsymbol{\omega} \equiv 0 \quad (435)$$

Transport The vorticity transport equation can be obtained from the curl of the momentum equation:

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\mathbf{u} - \boldsymbol{\omega}(\boldsymbol{\nabla}\cdot\mathbf{u}) + \boldsymbol{\nabla}T\times\boldsymbol{\nabla}s + \boldsymbol{\nabla}\times\left(\frac{\boldsymbol{\nabla}\cdot\boldsymbol{\tau}}{\rho}\right) \quad (436)$$

The cross products of the thermodynamic derivatives can be written as

$$\boldsymbol{\nabla}T\times\boldsymbol{\nabla}s = \boldsymbol{\nabla}P\times\boldsymbol{\nabla}v = -\frac{\boldsymbol{\nabla}P\times\boldsymbol{\nabla}\rho}{\rho^2} \quad (437)$$

which is known as the *baroclinic torque*.

For incompressible, homogeneous flow, the viscous term can be written $\nu\nabla^2\boldsymbol{\omega}$ and the incompressible vorticity transport equation for a homogeneous fluid is

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\mathbf{u} + \nu\nabla^2\boldsymbol{\omega} \tag{438}$$

Circulation The circulation Γ is defined as

$$\Gamma = \oint_{\partial\Omega} \mathbf{u}\cdot\,\mathrm{d}\mathbf{l} = \int_{\Omega} \boldsymbol{\omega}\cdot\hat{\mathbf{n}}\,\mathrm{d}A \tag{439}$$

where is Ω is a simple surface bounded by a closed curve $\partial\Omega$.

Vortex Lines and Tubes A *vortex line* is a curve drawn tangent to the vorticity vectors at each point in the flow.

$$\frac{\mathrm{d}x}{\omega_x} = \frac{\mathrm{d}y}{\omega_y} = \frac{\mathrm{d}z}{\omega_z} \tag{440}$$

The collection of vortex lines passing through a simple curve C form a vortex tube. On the surface of the vortex tube, we have $\hat{\mathbf{n}}\cdot\boldsymbol{\omega}$ =0.

A vortex tube of vanishing area is a *vortex filament*, which is characterized by a circulation Γ. The contribution $\mathrm{d}\mathbf{u}$ to the velocity field due to an element $\mathrm{d}\mathbf{l}$ of a vortex filament is given by the **Biot Savart** Law

$$\mathrm{d}\mathbf{u} = -\frac{\Gamma}{4\pi}\frac{\mathbf{r}\times\mathrm{d}\mathbf{l}}{r^3} \tag{441}$$

Line vortex A *potential vortex* has a singular vorticity field and purely azimuthal velocity field. For a single vortex located at the origin of a two-dimensional flow

$$\boldsymbol{\omega} = \hat{\mathbf{z}}\Gamma\delta(\mathbf{r}) \qquad u_\theta = \frac{\Gamma}{2\pi r} \tag{442}$$

For a line vortex of strength Γ_i located at (x_i, y_i), the velocity field at point (x, y) can be obtained by transforming the above result to get velocity components (u, v)

$$u = -\frac{\Gamma_i}{2\pi}\frac{y - y_i}{(x-x_i)^2 + (y-y_i)^2} \tag{443}$$

$$v = \frac{\Gamma_i}{2\pi}\frac{x - x_i}{(x-x_i)^2 + (y-y_i)^2} \tag{444}$$

$$\tag{445}$$

Or setting $\boldsymbol{\Gamma} = \hat{\mathbf{z}}\Gamma$

$$\mathbf{u}_i = \frac{\boldsymbol{\Gamma}_i\times\mathbf{r}_i}{2\pi r_i^2} \tag{446}$$

where $\mathbf{r}_i = \mathbf{i}$ - $\mathbf{x}_i$.

The *streamfunction* for the line vortex is found by integration to be

$$\psi_i = -\frac{\Gamma_i}{2\pi} \ln r_i \tag{447}$$

For a system of n vortices, the velocity field can be obtained by superposition of the individual contributions to the velocity from each vortex. In the absence of boundaries or other surfaces:

$$\mathbf{u} = \sum_{i=1}^{n} \frac{\mathbf{\Gamma}_i \times \mathbf{r}_i}{2\pi r_i^2} \tag{448}$$

4.7 Key Ideas about Vorticity

1. Vorticity can be visualized as local rotation within the fluid. The local angular frequency of rotation about the direction $\hat{\mathbf{n}}$ is

$$f_{\hat{\mathbf{n}}} = \lim_{r \to 0} \frac{u_\theta}{2\pi r} = \frac{1}{2\pi} \frac{|\boldsymbol{\omega} \cdot \hat{\mathbf{n}}|}{2}$$

2. Vorticity cannot begin or end within the fluid.

$$\boldsymbol{\nabla} \cdot \boldsymbol{\omega} = 0$$

3. The circulation is constant along a vortex tube or filament at a given instant in time

$$\int_{tube} \boldsymbol{\omega} \cdot \hat{\mathbf{n}}\, \mathrm{d}A = \text{constant}$$

However, the circulation can change with time due to viscous forces, baroclinic torque or nonconservative external forces. A vortex tube does not have a fixed identity in a time-dependent flow.

4. *Thompson's or Kelvin's theorem* Vortex filaments move with the fluid and the circulation is constant for an inviscid, homogeneous fluid subject only to conservative body forces.

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = 0 \tag{449}$$

Bjerknes theorem If the fluid is inviscid but inhomgeneous, $\rho(\mathbf{x}, t)$, then the circulation will change due to the baroclinic torque $\boldsymbol{\nabla} P \times \boldsymbol{\nabla} \rho$:

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = -\oint_{\partial\Omega} \frac{\mathrm{d}P}{\rho} = -\int_{\Omega} \frac{\boldsymbol{\nabla} P \times \boldsymbol{\nabla} \rho}{\rho^2} \cdot \hat{\mathbf{n}} \mathrm{d}A \tag{450}$$

Viscous fluids have an additional contribution due to the diffusion of vorticity into or out of the tube.

4.8 Unsteady Potential Flow

Bernoulli's equation for unsteady potential flow

$$P - P_\infty = -\rho \frac{\partial}{\partial t} (\phi - \phi_\infty) + \rho \frac{U_\infty^2}{2} - \rho \frac{|\boldsymbol{\nabla} \phi|^2}{2} \tag{451}$$

Induced Mass If the external force $\mathbf{F}_{ext}$ is applied to a body of mass M, then the acceleration of the body $d\mathbf{U}/dt$ is determined by

$$\mathbf{F}_{ext} = (m + \mathsf{M} \cdot) \frac{\mathrm{d}\mathbf{U}}{\mathrm{d}\mathbf{t}} \tag{452}$$

where M is the *induced mass tensor*. For a sphere (3-D) or a cylinder (2-D), the induced mass is simply $\mathsf{M} = m_i \mathsf{I}$.

$$m_{i,sphere} = \frac{2}{3} \pi a^3 \rho \tag{453}$$

$$m_{i,cylinder} = \pi a^2 \rho \tag{454}$$

$$\tag{455}$$

Bubble Oscillations The motion of a bubble of gas within an incompressible fluid can be described by unsteady potential flow in the limit of small-amplitude, low-frequency oscillations. The potential is given by the 3-D source solution. For a bubble of radius $R(t)$, the potential is

$$\phi = -\frac{R^2(t)}{r} \frac{\mathrm{d}R}{\mathrm{d}t} \tag{456}$$

Integration of the momentum equation in spherical coordinates yields the *Rayleigh* equation

$$R \frac{\mathrm{d}^2 R}{\mathrm{d}t^2} + \frac{3}{2} \left(\frac{\mathrm{d}R}{\mathrm{d}t} \right)^2 = \frac{P(R) - P_\infty}{\rho} \tag{457}$$

4.9 Complex Variable Methods

Two dimensional potential flow problems can be solved in the complex plane

$$z = x + iy = r \exp(i\theta) = r \cos\theta + ir \sin\theta$$

The *complex potential* is defined as

$$F(z) = \phi + i\psi \tag{458}$$

and the complex velocity w is defined as

$$w = u - iv = \frac{\mathrm{d}F}{\mathrm{d}z} \tag{459}$$

NB sign of v-term! The complex potential is an analytic function and the derivatives satisfy the *Cauchy-Riemann* conditions

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \tag{460}$$

$$\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{461}$$

which implies that both $\nabla^2 \phi = 0$ and $\nabla^2 \psi = 0$, i.e., the real and imaginary parts of an analytic function represent irrotational, potential flows.

Examples

1. Uniform flow $\mathbf{u} = (U_\infty, V_\infty)$

$$F = (U_\infty - iV_\infty)z$$

2. Line source of strength q located at z_o

$$F = \frac{q}{2\pi} \ln(z - z_\circ)$$

3. Line vortex of strength Γ located at $z_\circ$

$$F = -i\frac{\Gamma}{2\pi} \ln(z - z_\circ)$$

4. Source doublet (dipole) at $z_\circ$ oriented along $+x$ axis

$$F = -\frac{\mu}{2\pi(z - z_\circ)}$$

5. Vortex doublet at $z_\circ$ oriented along $+x$ axis

$$F = \frac{i\lambda}{2\pi(z - z_\circ)}$$

6. Stagnation point

$$F = Cz^2$$

7. Exterior corner flow

$$F = Cz^n \qquad 1/2 \leq n \leq 1$$

8. Interior corner flow, angle α

$$F = Cz^n \qquad 1 \leq n = \frac{\pi}{\alpha}$$

9. Circular cylinder at origin, radius a, uniform flow U at $x = \pm\infty$

$$F = U(z + \frac{a^2}{z})$$

4.9.1 Mapping Methods

A flow in the ζ plane can be mapped into the z plane using an analytic function $z = f(\zeta)$. An analytic function is a *conformal map*, preserving angles between geometric features such as streamlines and isopotentials as long as $\mathrm{d}f/\mathrm{d}z$ does not vanish. The velocity in the ζ-plane is $\tilde{w}$ and is related to the z-plane velocity by

$$\tilde{w} = \frac{\mathrm{d}F}{\mathrm{d}\zeta} = \frac{w}{\frac{\mathrm{d}\zeta}{\mathrm{d}z}} \qquad \text{or} \qquad w = \frac{\mathrm{d}F}{\mathrm{d}z} = \frac{\tilde{w}}{\frac{\mathrm{d}z}{\mathrm{d}\zeta}} \tag{462}$$

In order to have well behaved values of w, require $\tilde{w}$ =0 at point where $dz/d\zeta$ vanishes.

Blasius' Theorem The force on a cylindrical (2-D) body in a potential flow is given by

$$D - iL = \frac{i}{2}\rho \oint_{body} w^2 \,\mathrm{d}z \tag{463}$$

For rigid bodies

$$D = 0 \qquad L = -\rho U_\infty \Gamma \tag{464}$$

where the lift is perpendicular to the direction of fluid motion at ∞. The moment of force about the origin is

$$M = -\frac{1}{2}\rho \Re \left(\oint_{body} z w^2 \,\mathrm{d}z \right) \tag{465}$$

4.10 Airfoil Theory

Rotating Cylinder The streamfunction for a uniform flow U_∞ over a cylinder of radius a with a bound vortex of strength Γ is

$$\psi = U_\infty r \sin\theta \left[1 - \left(\frac{a}{r}\right)^2 \right] - \frac{\Gamma}{2\pi} \ln(\frac{r}{a}) \tag{466}$$

The stagnation points on the surface of the cylinder can be found at

$$\sin\theta_s = \frac{\Gamma}{4\pi U_\infty a} \tag{467}$$

The lift L is given by

$$L = -\rho U_\infty \Gamma \tag{468}$$

The pressure coefficient on the surface of the cylinder is

$$C_P = \frac{P - P_\infty}{\frac{1}{2}\rho U_\infty^2} = 1 - 4\sin^2\theta + \frac{4\Gamma}{2\pi a U_\infty}\sin\theta - \left(\frac{\Gamma}{2\pi a U_\infty}\right)^2 \tag{469}$$

Generalized Cylinder Flow If the flow at infinity is at angle α w.r.t. the x-axis, the complex potential for flow over a cylinder of radius a, center μ and bound circulation Γ is:

$$F(z) = U\left(\exp(-i\alpha)(z-\mu) + \frac{a^2\exp(i\alpha)}{z-\mu}\right) - i\frac{\Gamma}{2\pi}\ln\left(\frac{z-\mu}{a}\right) \tag{470}$$

Joukowski Transformation The transformation

$$z = \zeta + \frac{\zeta_T^2}{\zeta} \tag{471}$$

is the Joukowski transformation, which will map a cylinder of radius ζ_T in the ζ-plane to a line segment y =0, $-2\zeta_T \leq x \leq 2\zeta_T$. Use this together with the generalized cylinder flow in the ζ plane to produce the flow for a Joukowski arifoil at an angle of attack. The inverse transformation is

$$\zeta = \frac{z}{2} \pm \sqrt{\left(\frac{z}{2}\right)^2 - \zeta_T^2} \tag{472}$$

Kutta Condition The flow at the trailing edge of an airfoil must leave smoothly without any singularities. There are two special cases:

- For a finite-angle trailing edge in potential flow, the trailing edge must be a stagnation point.
- For a cusp (zero angle) trailing edge in potential flow, the velocity can be finite but must be equal on the two sides of the separating streamline.

Application to Joukowski airfoil: Locating the stagnation point at $\zeta_T = \mu + a\exp -i\beta$, the circulation is determined to be:

$$\Gamma_\kappa = -4\pi a U_\infty \sin(\alpha + \beta) \tag{473}$$

and the lift coefficient is

$$C_L = \frac{L}{\frac{1}{2}\rho U_\infty^2 c} = 8\pi\left(\frac{a}{c}\right)\sin(\alpha+\beta) \tag{474}$$

4.11 Thin-Wing Theory

The flow consists of the superposition of the free stream flow and an irrotational velocity field derived from disturbance potentials ϕ_t and ϕ_c associated with the thickness and camber functions.

$$u = U_\infty \cos\alpha + u_t + u_c \tag{475}$$
$$v = U_\infty \sin\alpha + v_t + v_c \tag{476}$$
$$\mathbf{u}_t = \nabla\phi_t \tag{477}$$
$$\mathbf{u}_c = \nabla\phi_c \tag{478}$$
$$\tag{479}$$

where α is the angle of attack and $\nabla^2\phi_i = 0$.

Geometry A thin, two-dimensional, wing-like body can be represented by two surfaces displaced slightly about a wing chord aligned with the x-axis, $0 \le x \le c$. The upper (+) and lower (−) surfaces of the wing are given by

$$y = Y_+(x) \quad \text{for} \quad \text{upper surface} \quad 0 \le x \le c \tag{480}$$
$$y = Y_-(x) \quad \text{for} \quad \text{lower surface} \quad 0 \le x \le c \tag{481}$$

and can be represented by a *thickness function* $f(x)$ and a *camber function* $g(x)$.

$$f(x) = Y_+(x) - Y_-(x) \tag{482}$$
$$g(x) = \frac{1}{2}\left[Y_+(x) + Y_-(x)\right] \tag{483}$$

The profiles of the upper and lower surface can be expressed in terms of f and g as

$$Y_+(x) = g(x) + \frac{1}{2}f(x) \quad \text{upper surface} \tag{484}$$
$$Y_-(x) = g(x) - \frac{1}{2}f(x) \quad \text{lower surface} \tag{485}$$

The maximum thickness $t = O(f)$, the maximum camber $h = O(g)$, and the angle of attack are all considered to be small in this analysis

$$\alpha \sim \frac{t}{c} \sim \frac{h}{c} \ll 1 \quad \text{and} \quad u_i, v_i << U_\infty \tag{486}$$

Boundary Conditions The exact slip boundary condition for inviscid flow on the body is:

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = \frac{v}{u}\bigg|_{(x,Y(x))} \tag{487}$$

The linearized version of this is:

$$\frac{\mathrm{d}Y_\pm}{\mathrm{d}x} = \alpha + \lim_{y\to\pm 0} \frac{v_t + v_c}{U_\infty}\bigg|_{(x,y)} \tag{488}$$

with $\cos\alpha \approx 1$, and $\sin\alpha \approx \alpha$. This can be written as

$$v_t(x,0+) + v_c(x,0+) = U_\infty\left(g' + \frac{1}{2}f'\right) - \alpha U_\infty \tag{489}$$

$$v_t(x,0-) + v_c(x,0-) = U_\infty\left(g' - \frac{1}{2}f'\right) - \alpha U_\infty \tag{490}$$

where $f' = \mathrm{d}f/\mathrm{d}x$ and $g' = \mathrm{d}g/\mathrm{d}x$.

The boundary conditions are then divided between the thickness and camber disturbance flows as follows:

$$v_t = \pm\frac{1}{2}U_\infty f' \quad \text{for} \quad y \to \pm 0 \tag{491}$$

$$v_c = U_\infty(g' - \alpha) \quad \text{for} \quad y \to \pm 0 \tag{492}$$

In addition, the disturbance velocities have to vanish far from the body.

4.11.1 Thickness Solution

The potential ϕ_t for the pure thickness case, which can be interpreted as a symmetric body at zero angle of attack, can be calculated by the superposition of sources of strength $q\,\mathrm{d}x$ using the general solution for potential flow

$$\phi_t(x,y) = \frac{1}{2\pi}\int_0^c \ln(y^2 + (x-\xi)^2)q(\xi)\,\mathrm{d}\xi \tag{493}$$

The velocity components are:

$$u_t = \frac{1}{2\pi}\int_0^c \frac{(x-\xi)q(\xi)\,\mathrm{d}\xi}{y^2 + (x-\xi)^2} \tag{494}$$

$$v_t = \frac{1}{2\pi}\int_0^c \frac{yq(\xi)\,\mathrm{d}\xi}{y^2 + (x-\xi)^2} \tag{495}$$

Apply the linearized boundary condition to obtain

$$\pm\frac{1}{2}U_\infty\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{y\to\pm 0}\frac{1}{2\pi}\int_0^c \frac{yq(\xi)\,\mathrm{d}\xi}{y^2 + (x-\xi)^2} \tag{496}$$

Delta Function Representation The limit of the integrand is one of the representations of the *Dirac delta function*

$$\lim_{y\to\pm 0}\frac{1}{\pi}\frac{y}{y^2+(x-\xi)^2}=\pm\delta(x-\xi) \tag{497}$$

where

$$\delta(x-\xi)=\begin{cases}0 & x\neq\xi\\ \infty & x=\xi\end{cases} \qquad \int_{-\infty}^{+\infty} f(\xi)\delta(x-\xi)\,\mathrm{d}\xi=f(x) \tag{498}$$

Source Distribution This leads to the source distribution

$$q(x)=U_\infty\frac{\mathrm{d}f}{\mathrm{d}x} \tag{499}$$

and the solution for the velocity field is

$$u_t = \frac{\partial\phi_t}{\partial x}=\frac{U_\infty}{2\pi}\int_0^c\frac{(x-\xi)f'(\xi)\,\mathrm{d}\xi}{y^2+(x-\xi)^2} \tag{500}$$

$$v_t = \frac{\partial\phi_t}{\partial y}=\frac{U_\infty}{2\pi}\int_0^c\frac{yf'(\xi)\,\mathrm{d}\xi}{y^2+(x-\xi)^2} \tag{501}$$

The velocity components satisfy the following relationships across the surface of the wing

$$[u] = u(x,0+)-u(x,0-)=0 \tag{502}$$

$$[v] = v(x,0+)-v(x,0-)=q(x) \tag{503}$$

Pressure Coefficient The pressure coefficient is defined to be

$$C_P=\frac{P-P_\infty}{\frac{1}{2}\rho U_\infty^2}=1-\frac{u^2+v^2}{U_\infty^2} \tag{504}$$

The linearized version of this is:

$$C_P\approx -2\frac{u_t+u_c}{U_\infty} \tag{505}$$

For the pure thickness case, then we have the following result:

$$C_P\approx -\frac{1}{\pi}\int_0^c\frac{f'(\xi)\,\mathrm{d}\xi}{(x-\xi)} \tag{506}$$

The integral is to be evaluated in the sense of the *Principal value* interpretation.

Principal Value Integrals If an integral has an integrand g that is singular at $\xi = x$, the principal value or finite part is defined as

$$\mathsf{P}\int_0^a g(\xi)\,\mathrm{d}\xi = \lim_{\epsilon\to 0}\left[\int_0^{x-\epsilon} g(\xi)\,\mathrm{d}\xi + \int_{x+\epsilon}^a g(\xi)\,\mathrm{d}\xi\right] \tag{507}$$

Important principal value integrals are

$$\mathsf{P}\int_0^c \frac{\mathrm{d}\xi}{(x-\xi)} = \ln\left(\frac{x}{x-c}\right) \tag{508}$$

and

$$\mathsf{P}\int_0^c \frac{\xi^{-1/2}\mathrm{d}\xi}{(x-\xi)} = \frac{1}{\sqrt{x}}\ln\left(\frac{\sqrt{c}+\sqrt{x}}{\sqrt{c}-\sqrt{x}}\right) \tag{509}$$

A generalization to other powers can be obtained by the recursion relation

$$\mathsf{P}\int_0^c \frac{\xi^n\mathrm{d}\xi}{(x-\xi)} = x\mathsf{P}\int_0^c \frac{\xi^{n-1}\mathrm{d}\xi}{(x-\xi)} - \frac{c^n}{n} \tag{510}$$

A special case can be found for the transformed variables $\cos\theta = 1$ - $2\xi/c$

$$\mathsf{P}\int_0^\pi \frac{\cos n\theta\mathrm{d}\theta}{\cos\theta - \cos\theta_o} = \pi\frac{\sin n\theta_o}{\sin\theta_o} \tag{511}$$

4.11.2 Camber Case

The camber case alone accounts for the lift (non-zero α) and the camber. The potential ϕ_c for the pure camber case can be represented as a superposition of potential vortices of strength $\gamma(x)\,\mathrm{d}x$ along the chord of the wing:

$$\phi_c = \frac{1}{2\pi}\int_0^c \gamma(\xi)\tan^{-1}\left(\frac{y}{x-\xi}\right)\mathrm{d}\xi \tag{512}$$

The velocity components are:

$$u_c = \frac{\partial\phi_c}{\partial x} = -\frac{1}{2\pi}\int_0^c \frac{y\gamma(\xi)\,\mathrm{d}\xi}{y^2+(x-\xi)^2} \tag{513}$$

$$v_c = \frac{\partial\phi_c}{\partial y} = \frac{1}{2\pi}\int_0^c \frac{(x-\xi)\gamma(\xi)\,\mathrm{d}\xi}{y^2+(x-\xi)^2} \tag{514}$$

The u component of velocity on the surface of the wing is

$$\lim_{y\to\pm 0} u_c(x,y) = u(x,\pm 0) = \mp\frac{\gamma(x)}{2} \tag{515}$$

Apply the linearized boundary condition to obtain the following integral equation for the vorticity distribution γ

$$U_\infty \left(\frac{\mathrm{d}g}{\mathrm{d}x} - \alpha \right) = \frac{1}{2\pi} \mathsf{P} \int_0^c \frac{\gamma(\xi)\,\mathrm{d}\xi}{(x-\xi)}\,\mathrm{d}\xi \tag{516}$$

The total circulation Γ is given by

$$\Gamma = \int_0^c \gamma(\xi)\,\mathrm{d}\xi \tag{517}$$

The velocity components satisfy the following relationships across the surface of the wing

$$[u] = u(x,0+) - u(x,0-) = -\gamma(x) \tag{518}$$

$$[v] = v(x,0+) - v(x,0-) = 0 \tag{519}$$

Kutta Condition The Kutta condition at the trailing edge of a sharp-edged airfoil reduces to

$$\gamma(x=c) = 0 \tag{520}$$

Vorticity Distribution The integral equation for the vorticity distribution can be solved explicity. A solution that satisfies the Kutta boundary condition is:

$$\gamma(x) = -2U_\infty \left(\frac{c-x}{x} \right)^{1/2} \left[\alpha + \frac{1}{\pi} \mathsf{P} \int_0^c \frac{g'(\xi)}{x-\xi} \left(\frac{\xi}{c-\xi} \right)^{1/2} \mathrm{d}\xi \right] \tag{521}$$

The pressure coefficient for the pure camber case is

$$C_P = \pm \frac{\gamma(x)}{U_\infty} \quad \text{for} \quad y \to \pm 0 \tag{522}$$

The integrals can be computed exactly for several special cases, which can be expressed most conveniently using the transformation

$$z = \frac{2x}{c} - 1 \qquad \rho = \frac{2\xi}{c} - 1 \tag{523}$$

$$\mathsf{P} \int_{-1}^{1} \frac{1}{z-\rho} \sqrt{\frac{1+\rho}{1-\rho}}\,\mathrm{d}\rho = -\pi \tag{524}$$

$$\mathsf{P} \int_{-1}^{1} \frac{\sqrt{1-\rho^2}}{z-\rho}\,\mathrm{d}\rho = \pi z \tag{525}$$

$$\mathsf{P} \int_{-1}^{1} \frac{1}{\sqrt{1-\rho^2}(z-\rho)}\,\mathrm{d}\rho = 0 \tag{526}$$

$$\mathsf{P} \int_{-1}^{1} \frac{\rho}{\sqrt{1-\rho^2}(z-\rho)}\,\mathrm{d}\rho = -\pi \tag{527}$$

$$\mathsf{P}\int_{-1}^{1} \frac{\rho^2}{\sqrt{1-\rho^2}(z-\rho)}\,\mathrm{d}\rho = -\pi z \tag{528}$$

Higher powers of the numerator can be evaluated from the recursion relation:

$$\mathsf{P}\int_{-1}^{1} \frac{\rho^n}{\sqrt{1-\rho^2}(z-\rho)}\,\mathrm{d}\rho = z\mathsf{P}\int_{-1}^{1} \frac{\rho^{n-1}}{\sqrt{1-\rho^2}(z-\rho)}\,\mathrm{d}\rho - \frac{\pi}{2}\left[1-(-1)^n\right]\frac{1(3)\cdots(n-2)}{2(4)\cdots(n-1)} \tag{529}$$

4.12 Axisymmetric Slender Bodies

Disturbance potential solution using source distribution on x-axis:

$$\phi(x,r) = -\frac{1}{4\pi}\int_0^c \frac{f(\xi)\,\mathrm{d}\xi}{\sqrt{(x-\xi)^2+r^2}} \tag{530}$$

Velocity components:

$$u = U_\infty + \frac{\partial\phi}{\partial x} = \frac{1}{4\pi}\int_0^c \frac{(x-\xi)f(\xi)\,\mathrm{d}\xi}{\left[(x-\xi)^2+r^2\right]^{3/2}} \tag{531}$$

$$v = \frac{\partial\phi}{\partial r} = \frac{1}{4\pi}\int_0^c \frac{rf(\xi)\,\mathrm{d}\xi}{\left[(x-\xi)^2+r^2\right]^{3/2}} \tag{532}$$

$$\tag{533}$$

Exact boundary condition on body $R(x)$

$$\left.\frac{v}{u}\right|_{(x,R(x))} = \frac{\mathrm{d}R}{\mathrm{d}x} \tag{534}$$

Linearized boundary condition, first approximation:

$$v(x, r=R) = U_\infty\frac{\mathrm{d}R}{\mathrm{d}x} \tag{535}$$

Extrapolation to x axis:

$$\lim_{r\to 0}(2\pi r v) = 2\pi R\frac{\mathrm{d}R}{\mathrm{d}x}U_\infty \tag{536}$$

Source strength

$$f(x) = U_\infty 2\pi R\frac{\mathrm{d}R}{\mathrm{d}x} = U_\infty A'(x) \qquad A(x) = \pi R^2(x) \tag{537}$$

Pressure coefficient

$$C_P \approx -\frac{2u}{U_\infty} - \left(\frac{\mathrm{d}R}{\mathrm{d}x}\right)^2 \tag{538}$$

4.13 Wing Theory

Wing span is $-b/2 < y < +b/2$. The section lift coefficient, L' = lift per unit span

$$C'_L(y) = \frac{L'}{\frac{1}{2}\rho U_\infty^2 c(y)} = m_\circ(y)\,(\alpha - \alpha_i - \alpha_\circ(y)) \tag{539}$$

Induced angle of attack, w = downwash velocity

$$\alpha_i = \tan^{-1}\left(\frac{w}{U_\infty}\right) \approx \frac{w}{U_\infty} \tag{540}$$

Induced drag

$$D_i = \rho U_\infty \Gamma \alpha_i \tag{541}$$

Load distribution $\Gamma(y)$, bound circulation at span location y

$$\Gamma(y) = \frac{1}{2} m_\circ U_\infty c(y)\,(\alpha - \alpha_i - \alpha_\circ(y)) \tag{542}$$

Trailing vortex sheet strength

$$\gamma = -\frac{\mathrm{d}\Gamma}{\mathrm{d}y} \tag{543}$$

Downwash velocity

$$w = \frac{1}{4pi}\mathsf{P}\int_{-b/2}^{+b/2} \frac{\gamma(\xi)\,\mathrm{d}\xi}{\xi - y} \tag{544}$$

Integral equation for load distribution

$$\Gamma(y) = \frac{1}{2} m_\circ(y) U_\infty c(y)\left[\alpha - \alpha_\circ(y) - \frac{1}{4piU_\infty}\mathsf{P}\int_{-b/2}^{+b/2} \frac{\Gamma'(\xi)\,\mathrm{d}\xi}{\xi - y}\right] \tag{545}$$

Boundary conditions

$$\Gamma(\frac{b}{2}) = \Gamma(-\frac{b}{2}) = 0 \tag{546}$$

Elliptic load distribution, constant downwash, induced angle of attack

$$\Gamma(y) = \Gamma_s\left[1 - \left(\frac{y}{2b}\right)^2\right]^{1/2} \qquad w = \frac{\Gamma_s}{2b} \qquad \alpha_i = \frac{\Gamma_s}{2U_\infty} \tag{547}$$

Lift

$$L = \rho U_\infty \Gamma_s \frac{\pi b^2}{4} \tag{548}$$

Induced drag (minimum for elliptic loading)

$$D_i = \frac{1}{\pi} \frac{L^2}{\frac{1}{2}\rho U_\infty^2 b^2} \tag{549}$$

Induced drag coefficient

$$C_{D,i} = \frac{C_L^2}{\pi AR} \qquad AR = b^2/S \approx \frac{b}{c} \tag{550}$$

5 Viscous Flow

Equations of motion in cartesian tensor form (without body forces) are:
Conservation of mass:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_k}{\partial x_k} = 0 \tag{551}$$

Momentum equation:

$$\rho \frac{\partial u_i}{\partial t} + \rho u_k \frac{\partial u_i}{\partial x_k} = -\frac{\partial P}{\partial x_i} + \frac{\partial \tau_{ik}}{\partial x_k} \quad (i = 1, 2, 3) \tag{552}$$

Viscous stress tensor

$$\tau_{ik} = \mu \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) + \lambda \delta_{ik} \frac{\partial u_j}{\partial x_j} \qquad \text{sum on j} \tag{553}$$

Lamé's constant

$$\lambda = \mu_v - \frac{2}{3}\mu \tag{554}$$

Energy equation, total enthalpy form:

$$\rho \frac{\partial h_t}{\partial t} + \rho u_k \frac{\partial h_t}{\partial x_k} = \frac{\partial P}{\partial t} + \frac{\partial \tau_{ki} u_i}{\partial x_k} - \frac{\partial q_i}{\partial x_i} \qquad \text{sum on i and k} \tag{555}$$

Thermal energy form

$$\rho \frac{\partial h}{\partial t} + \rho u_k \frac{\partial h}{\partial x_k} = \frac{\partial P}{\partial t} + u_k \frac{\partial P}{\partial x_k} + \tau_{ik} \frac{\partial u_i}{\partial x_k} - \frac{\partial q_i}{\partial x_i} \qquad \text{sum on i and k} \tag{556}$$

or alternatively

$$\rho \frac{\partial e}{\partial t} + \rho u_k \frac{\partial e}{\partial x_k} = -P \frac{\partial u_k}{\partial x_k} + \tau_{ik} \frac{\partial u_i}{\partial x_k} - \frac{\partial q_i}{\partial x_i} \qquad \text{sum on i and k} \tag{557}$$

Dissipation function

$$\Upsilon = \tau_{ik} \frac{\partial u_i}{\partial x_k} \tag{558}$$

Fourier's law

$$q_i = -k \frac{\partial T}{\partial x_i} \tag{559}$$

5.1 Scaling

Reference conditions are

velocity	$U_\circ$
length	L
time	T
density	$\rho_\circ$
viscosity	$\mu_\circ$
thermal conductivity	$k_\circ$

Inertial flow Limit of vanishing viscosity, $\mu \to 0$. Nondimensional statement:

$$\text{Reynolds number} \quad Re = \frac{\rho_\circ U_\circ L}{\mu_\circ} \gg 1 \quad P \sim \rho_\circ U_\circ^2 \tag{560}$$

Nondimensional momentum equation

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\boldsymbol{\nabla} P + \frac{1}{Re} \boldsymbol{\nabla} \cdot \boldsymbol{\tau} \tag{561}$$

Limiting case, $Re \to \infty$, inviscid flow

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\boldsymbol{\nabla} P \tag{562}$$

Viscous flow Limit of vanishing density, $\rho \to 0$. Nondimensional statement:

$$\text{Reynolds number} \qquad Re = \frac{\rho_\circ U_\circ L}{\mu_\circ} \ll 1 \qquad P \sim \frac{\mu_\circ U_\circ}{L} \tag{563}$$

Nondimensional momentum equation

$$Re \rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\boldsymbol{\nabla} P + \boldsymbol{\nabla} \cdot \boldsymbol{\tau} \tag{564}$$

Limiting case, $Re \to 0$, creeping flow.

$$\boldsymbol{\nabla} P = \boldsymbol{\nabla} \cdot \boldsymbol{\tau} \tag{565}$$

5.2 Two-Dimensional Flow

For a viscous flow in two-space dimensions (x, y) the components of the viscous stress tensor in cartersian coordinates are

$$\boldsymbol{\tau} = \begin{pmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{yx} & \tau_{yy} \end{pmatrix} = \begin{pmatrix} 2\mu \frac{\partial u}{\partial x} + \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) & \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\ \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) & 2\mu \frac{\partial v}{\partial y} + \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \end{pmatrix} \tag{566}$$

Dissipation function

$$\Upsilon = \mu \left[2 \left(\frac{\partial u}{\partial x} \right)^2 + 2 \left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \right] + \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)^2 \tag{567}$$

5.3 Parallel Flow

The simplest case of viscous flow is parallel flow,

$$(u, v) = (u(y, t), 0) \Rightarrow \boldsymbol{\nabla} \cdot \mathbf{u} = 0 \Rightarrow \rho = \rho(y) \quad \text{only} \tag{568}$$

Momentum equation

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \tag{569}$$

$$0 = -\frac{\partial P}{\partial y} \tag{570}$$

We conclude from the y-momentum equation that $P = P(x)$ only.
Energy equation

$$\rho \frac{\partial e}{\partial t} + \rho u \frac{\partial e}{\partial x} = \mu \left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) \tag{571}$$

5.3.1 Steady Flows

In these flows $\frac{\partial}{\partial t} = 0$ and inertia plays no role. Shear stress is either constant or varies only due to imposed axial pressure gradients.

Couette Flow A special case are flows in which pressure gradients are absent

$$\frac{\partial P}{\partial x} = 0 \tag{572}$$

and the properties strictly depend only on the y coordinate, these flows have $\frac{\partial}{\partial x} = 0$. The shear stress is constant in these flows

$$\tau_{xy} = \mu \frac{\partial u}{\partial y} = \tau_w \tag{573}$$

The motion is produced by friction at the moving boundaries

$$u(y = H) = U \qquad u(y = 0) = 0 \tag{574}$$

and given the viscosity $\mu(y)$ the velocity profile and shear stress τ_w can be determined by integration

$$u(y) = \tau_w \int_o^y \frac{\mathrm{d}y'}{\mu(y')} \qquad \tau_w = U \left(\int_0^H \frac{\mathrm{d}y'}{\mu(y')}\right)^{-1} \tag{575}$$

The dissipation is balanced by thermal conduction in the y direction.

$$\mu \left(\frac{\partial u}{\partial y}\right)^2 = -\frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) \tag{576}$$

Using the constant shear stress condition, we have the following energy integral

$$u\tau - q = -q_w = \text{constant} \qquad q = -k\frac{\partial T}{\partial y} \qquad q_w = q(y=0) \tag{577}$$

This relationship can be further investigated by defining the *Prandtl* number

$$Pr = \frac{c_P\mu}{k} = \frac{\nu}{\kappa} \qquad \kappa = \frac{k}{\rho c_P} \tag{578}$$

For gases, $Pr \sim 0.7$, approximately independent of temperature. The Eucken relation is a useful approximation that only depends on the ratio of specific heats γ

$$Pr \approx \frac{4\gamma}{7.08\gamma - 1.80} \tag{579}$$

For many gases, both viscosity and conductivity can be approximated by power laws $\mu \sim T^n$, $k \sim T^m$ where the exponents n and m range between 0.65 to 1.4 depending on the substance.

Constant Prandtl Number Assuming $Pr =$ constant and using $\mathrm{d}\,h = c_p \mathrm{d}\,T$, the energy equation can be integrated to obtain the *Crocco-Busemann* relation

$$h - h_w + Pr\frac{u^2}{2} = -\frac{q_w}{\tau_w} Pr\, u \tag{580}$$

For constant c_P, this is

$$T = T_w - Pr\frac{u^2}{2c_P} - \frac{q_w}{\tau_w}\frac{Pr}{c_P}u \tag{581}$$

Recovery Temperature If the lower wall ($y = 0$) is insulated $q_w = 0$, then the temperature at $y = 0$ is defined to be the *recovery temperature*. In terms of the conditions at the upper plate ($y = H$), this defines a recovery enthalpy

$$h_r = h(T_r) \equiv h(T_H) + Pr\frac{1}{2}U^2 \tag{582}$$

If the heat capacity $c_P =$ constant and we use the conventional boundary layer notation, for which $T_H = T_e$, the temperature at the outer edge of the boundary layer

$$T_r = T_e + Pr\frac{1}{2}\frac{U^2}{c_P} \tag{583}$$

Contrast with the adiabatic stagnation temperature

$$T_t = T_e + \frac{1}{2}\frac{U^2}{c_P} \tag{584}$$

The *recovery factor* is defined as

$$r = \frac{T_r - T_e}{T_t - T_e} \tag{585}$$

In Couette flow, $r = Pr$. The wall temperature is lower than the adiabatic stagnation temperature T_t when $Pr < 1$, due to thermal conduction removing energy faster than it is being generated by viscous dissipation. If $Pr > 1$, then viscous dissipation generates heat faster than it can be conducted away from the wall and $T_r > T_t$.

Reynolds Analogy If the wall is not adiabatic, then the heat flux at the lower wall may significantly change the temperature profile. In particular the lower wall temperature (for c_p = constant) is

$$T_w = T_r + \frac{q_w}{c_P \tau_w} PrU \tag{586}$$

In order to heat the fluid $q_w > 0$, the lower wall must be *hotter than the recovery temperature.*

The heat transfer from the wall can be expressed as a heat transfer coefficient or *Stanton* number

$$St = \frac{q_w}{\rho U c_P (T_w - T_r)} \tag{587}$$

where q_w is the heat flux *from the wall into the fluid*, which is positive when heat is being added to the fluid. The Stanton number is proportional to the *skin friction* coefficient

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} \tag{588}$$

For Couette flow,

$$St = \frac{C_f}{2Pr} \tag{589}$$

This relationship between skin friction and heat transfer is the *Reynolds analogy.*

Constant properties If μ and k are constant, then the velocity profile is linear:

$$\tau_w = \mu \frac{U}{H} \qquad u = \frac{\tau_w}{\mu} y \tag{590}$$

The skin friction coefficient is

$$C_f = \frac{2}{Re} \qquad Re = \frac{\rho U H}{\mu} \tag{591}$$

5.3.2 Poiseuille Flow

If an axial pressure gradient is present, $\frac{\partial P}{\partial x} < 0$, then the shear stress will vary across the channel and fluid motion will result even when the walls are stationary. In that case, the shear stress balances the pressure drop. This is the usual situation in industrial pipe and channel flows. For the simple case of constant μ

$$0 = -\frac{\partial P}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} \tag{592}$$

With the boundary conditions $u(0) = u(H) = 0$, this can be integrated to yield the velocity distribution

$$u = -\frac{\partial P}{\partial x}\frac{H^2}{2\mu}\frac{y}{H}\left(1 - \frac{y}{H}\right) \tag{593}$$

and the wall shear stress

$$\tau_w = -\frac{\partial P}{\partial x}\frac{H}{2} \tag{594}$$

Pipe Flow The same situation for a round channel, a pipe of radius R, reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u}{\partial r} = \frac{1}{\mu}\frac{\partial P}{\partial x} \tag{595}$$

which integrates to the velocity distribution

$$u = -\frac{1}{4\mu}\frac{\partial P}{\partial x}\left(R^2 - r^2\right) \tag{596}$$

and a wall shear stress of

$$\tau_w = -\frac{\partial P}{\partial x}\frac{R}{2} \tag{597}$$

The total volume flow rate is

$$Q = -\frac{\partial P}{\partial x}\frac{\pi R^4}{8\mu} \tag{598}$$

The skin friction coefficient is traditionally based on the mean speed $\bar{u}$ and using the pipe diameter $d = 2R$ as the scale length.

$$\bar{u} = \frac{Q}{\pi R^2} = -\frac{\partial P}{\partial x}\frac{R^2}{8\mu} \tag{599}$$

and is equal to

$$C_f = \frac{\tau_w}{1/2\rho\bar{u}^2} = \frac{16}{Re_d} \qquad Re_d = \frac{\rho\bar{u}d}{\mu} \tag{600}$$

In terms of the *Darcy friction factor*,

$$\Lambda = \frac{8\tau_w}{\rho\bar{u}^2} = \frac{64}{Re_d} \tag{601}$$

Turbulent flow in smooth pipes is correlated by *Prandtl's formula*

$$\frac{1}{\sqrt{\Lambda}} = 2.0\log\left(Re_d\sqrt{\Lambda}\right) - 0.8 \tag{602}$$

or the simpler curvefit

$$\Lambda = 1.02\left(\log Re_d\right)^{-2.5} \tag{603}$$

5.3.3 Rayleigh Problem

Also known as *Stokes' first problem.* Another variant of parallel flow is *unsteady* flow with no gradients in the x direction. The Rayleigh problem is to determine the motion above an infinite ($-\infty < x < \infty$) plate impulsively accelerated parallel to itself.

The x-momentum equation (for constant μ) is

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \tag{604}$$

The boundary conditions are

$$u(y, t = 0) = 0 \qquad u(y = 0, t > 0) = U \tag{605}$$

The problem is *self similar* and in terms of the similarity variable η, the solution is

$$u = U f(\eta) \qquad \eta = \frac{y}{\sqrt{\nu t}} \qquad f'' + \frac{\eta}{2} f' = 0 \tag{606}$$

The solution is the *complementary error function*

$$f = \operatorname{erfc}(\frac{\eta}{2}) \quad \operatorname{erfc}(s) = 1 - \operatorname{erf}(s) \quad \operatorname{erf}(s) = \frac{2}{\sqrt{\pi}} \int_0^s \exp(-x^2)\, \mathrm{d}x \tag{607}$$

Shear stress at the wall

$$\tau_w = -\frac{\mu U}{\sqrt{\pi \nu t}} \tag{608}$$

Vorticity

$$\omega = -\frac{\partial u}{\partial y} = \frac{U}{\sqrt{\pi \nu t}} \exp(-\frac{\eta^2}{4}) \tag{609}$$

Vorticity thickness

$$\delta_\omega = \frac{1}{\omega_\circ} \int_0^\infty \omega(y, t)\, \mathrm{d}y = \sqrt{\pi \nu t} \tag{610}$$

5.4 Boundary Layers

For streamline bodies without separation, viscous effects are confined to a thin layer $y \leq \delta$, when the Reynolds number is sufficiently high, $Re \gg 1$.

Scaling

$$x \sim L \tag{611}$$

$$y \sim \delta \tag{612}$$

$$u \sim U \tag{613}$$

$$v \sim \frac{\delta}{L} U \sim \frac{U}{Re^{1/2}} \tag{614}$$

$$\delta \sim \frac{L}{Re^{1/2}} \tag{615}$$

Exterior or outer flow, u_e. $Re \to \infty$, slip boundary conditions. Equations are inviscid flow equations of motion.

Interior or inner flow , u_i. Finite Re but $\delta \ll L$, noslip boundary conditions $u_i(y=0) = 0$, matching to outer flow, $\lim_{y\to\infty} u_i = lim_{y\to 0} u_e$. Equations are

Boundary Layer Equations The unsteady, compressible boundary-layer equations are:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \tag{616}$$

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} = -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \tag{617}$$

$$0 = -\frac{\partial P}{\partial y} \tag{618}$$

$$\rho\frac{\partial h_t}{\partial t} + \rho u\frac{\partial h_t}{\partial x} + \rho v\frac{\partial h_t}{\partial y} = \frac{\partial P}{\partial t} + \frac{\partial}{\partial y}\left(u\tau_{xy} - q_y\right) \tag{619}$$

Thickness Measures 99% velocity thickness

$$\delta_{.99} = y(u = .99u_e) \tag{620}$$

Displacement thickness

$$\delta^* = \int_0^\infty \left(1 - \frac{\rho u}{\rho_e u_e}\right) \mathrm{d}y \tag{621}$$

Momentum thickness

$$\theta = \int_0^\infty \frac{\rho u}{\rho_e u_e}\left(1 - \frac{u}{u_e}\right) \mathrm{d}y \tag{622}$$

Displacement Velocity Near the boundary layer, the external flow produces a vertical velocity v_e which can be estimated by continuity to be

$$\rho_e v_e \approx -y\frac{\partial \rho_e u_e}{\partial x} \tag{623}$$

The boundary layer displaces the outer flow, producing a vertical velocity v far from the surface which differs from v_e by the amount v^*

$$\rho_e v* = \frac{\mathrm{d}}{\mathrm{d}x}\left(\rho_e u_e \delta^*\right) \tag{624}$$

The boundary layer influence on the outer flow can therefore by visualized as a source distribution producing an equivalent displacement.

Steady Incompressible Boundary layers The pressure gradient can be replaced by using Bernoulli's equation in the outer flow

$$\frac{\partial P}{\partial x} = -\rho u_e \left.\frac{\partial u_e}{\partial x}\right|_{y=0} \tag{625}$$

For constant μ and k, the equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{626}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_e\frac{\partial u_e}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{627}$$

$$\rho u\frac{\partial e}{\partial x} + \rho v\frac{\partial e}{\partial y} = \mu\left(\frac{\partial u}{\partial y}\right)^2 + k\frac{\partial^2 T}{\partial y^2} \tag{628}$$

5.4.1 Blasius Flow

The steady flow $u_e = U$ over a semi-infinite flat plate ($0 \leq x < \infty$) with no pressure gradient can be solved by a similarity transformation for the case of isothermal, incompressible flow.

$$\eta = \frac{y}{\delta(x)} \qquad \delta(x) = \sqrt{\frac{2\nu x}{U}} \tag{629}$$

Define a stream function

$$u = \frac{\partial \psi}{\partial y} \quad v = -\frac{\partial \psi}{\partial x} \quad \psi = \delta(x) U f(\eta) \tag{630}$$

to obtain the *Blasius* equation

$$f''' + ff'' = 0 \qquad f(0) = f'(0) = 0 \qquad f'(\infty) = 1 \tag{631}$$

Numerical solution yields $f''(0) = 0.469600$ for a skin friction coefficient of

$$C_f = \frac{0.664}{Re_x^{1/2}} \qquad Re_x = \frac{\rho U x}{\mu} \tag{632}$$

The various thickness measures are:

$$\delta_{.99} = \frac{5.0x}{Re_x^{1/2}} \qquad \delta^* = \frac{1.7208x}{Re_x^{1/2}} \qquad \theta = \frac{0.664x}{Re_x^{1/2}} \tag{633}$$

The displacement is equivalent to that produced by a slender body of thickness $\delta^*(x)$. The vertical velocity outside the boundary layer ($y \to \infty$) is

$$v^* \sim U\frac{\mathrm{d}\delta^*}{\mathrm{d}x} = \frac{0.861U}{Re_x^{1/2}} \tag{634}$$

which agrees with direct computation from the stream function

$$v^* = \lim_{\eta\to\infty} -\frac{\partial\psi}{\partial x} = \lim_{\eta\to\infty} \frac{U}{\sqrt{2}Re_x^{1/2}} \left(\eta f'(\eta) - f(\eta)\right) \tag{635}$$

where by numerical computation

$$\lim_{\eta\to\infty} f = \eta - \eta^* \qquad \eta^* = 1.21678 \qquad f'(\infty) = 1 \tag{636}$$

5.4.2 Falkner-Skan Flow

For flows of the type $u_e = Cx^m$, i.e., external flows representing flow over an exterior or interior corner of angle $\alpha = \pi m/(m+1)$, similarity solutions to the boundary layer equations can be obtained. Define the similarity variable and streamfunction similar to Blasius case

$$\eta = y/\delta(x) \quad \delta = \sqrt{\frac{2\nu x}{(m+1)u_e(x)}} \qquad \psi = u_e(x)\delta(x)f(\eta) \tag{637}$$

The resulting equation for the function f is

$$f''' + ff'' + \beta\left(1 - f'^2\right) = 0 \qquad \beta = \frac{2m}{m+1} \tag{638}$$

Some cases

m	flow
-.0904	separating
< 0	retarded flows, expansion corner
0	flat plate, zero pressure gradient
1	stagnation point
$0 <$	accelerated flows, wedges
-2	doublet near a wall
-1	point sink

5.5 Kármán Integral Relations

Integration of the momentum equation for incompressible flow results in

$$\frac{C_f}{2} = \frac{\mathrm{d}\theta}{\mathrm{d}x} + (2+H)\frac{\theta}{u_e}\frac{\mathrm{d}u_e}{\mathrm{d}x} \quad H = \frac{\delta^*}{\theta} \tag{639}$$

The *Kármán-Pohlhausen* technique consists of assuming a Blausius-type similarity profile for the velocity

$$u = u_e(x)f(\eta) \quad \eta = \frac{y}{\delta} \tag{640}$$

where δ locates a definite outer edge of the boundary layer. Matching the boundary layer solution smoothly to the outer flow at $\eta = 1$ and satisfying the noslip condition at $\eta = 0$, results in the following conditions on f

$$f(0) = 0 \tag{641}$$
$$f'(0) = \frac{\delta \tau_w}{\mu u_e} \tag{642}$$
$$f''(0) = -\frac{\delta^2}{\nu}\frac{\mathrm{d}u_e}{\mathrm{d}x} \tag{643}$$
$$f'''(0) = 0 \tag{644}$$
$$f(1) = 1 \tag{645}$$
$$f^{n>1}(1) = 0 \tag{646}$$

This results in an ordinary differential equation for δ as a function of x.

5.6 Thwaites' Method

Rewrite the Kármán integral equation as

$$\frac{u_e}{\nu}\frac{\mathrm{d}\theta^2}{\mathrm{d}x} = 2(S-(2+H)\lambda) \qquad \lambda = \frac{\theta^2}{\nu}\frac{\mathrm{d}u_e}{\mathrm{d}x} \qquad S = \frac{\theta\tau_w}{\mu u_e} \tag{647}$$

Thwaites' 1949 correlation

$$2(S-(H+2)\lambda) \approx 0.45 - 6\lambda \tag{648}$$

Kármán integral equation

$$u_e\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\lambda}{u_e'}\right) = 0.45 - 6\lambda \qquad u_e' = \frac{\mathrm{d}u_e}{\mathrm{d}x} \tag{649}$$

Approximate solution

$$\theta^2 = \frac{0.45\nu}{u_e^6}\int_0^x u_e^5\,\mathrm{d}x \tag{650}$$

Correlation functions $S(\lambda)$ and $H(\lambda)$

$$\tau_w = \frac{\mu u_e}{\theta}S(\lambda) \qquad \delta^* = \theta H(\lambda) \tag{651}$$

5.7 Laminar Separation

Seperation of the boundary layer from the wall and the creation of a recirculating flow region occurs when the shear stress vanishes.

$$\tau_{w,sep} = \mu\left.\frac{\partial u}{\partial y}\right|_{y=0,x=x_{sep}} = 0 \tag{652}$$

For laminar boundary layers, this occurs when a sufficiently long region of adverse pressure gradient $\mathrm{d}P/\mathrm{d}x > 0$ exists.

$$\frac{\delta_{.99}^2}{\mu u_e}\left.\frac{\mathrm{d}P}{\mathrm{d}x}\right|_{sep} \simeq 5 \qquad \lambda_{sep} \simeq -0.0931 \tag{653}$$

5.8 Compressible Boundary Layers

Steady, compressible, two-dimensional boundary layer equations:

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \tag{654}$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial P_e}{\partial x} + \frac{\partial}{\partial y}\mu \frac{\partial u}{\partial y} \tag{655}$$

$$\rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} = \mu \left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial}{\partial y} k \frac{\partial T}{\partial y} \tag{656}$$

5.8.1 Transformations and Approximations

Modified stream function

$$\rho u = \rho_\circ \frac{\partial \Psi}{\partial y} \qquad \rho v = -\rho_\circ \frac{\partial \Psi}{\partial x} \tag{657}$$

Density-weighted y-coordinate (Howarth-Doronitsyn-Stewartson)

$$Y = \int \frac{\rho}{\rho_\circ}\, \mathrm{d}y \qquad X = x \tag{658}$$

Derivative transformation

$$\frac{\partial}{\partial y} = \frac{\rho}{\rho_\circ}\frac{\partial}{\partial Y} \qquad \frac{\partial}{\partial x} = \frac{\partial}{\partial X} + \frac{\partial Y}{\partial x}\frac{\partial}{\partial Y} \tag{659}$$

Chapman-Rubesin parameter, enthalpy-temperature relation

$$C = \frac{\rho \mu}{\rho_\circ \mu_\circ} \qquad \mathrm{d}h = c_p\, \mathrm{d}T \tag{660}$$

Boundary layer equations

$$\frac{\partial \Psi}{\partial Y}\frac{\partial^2 \Psi}{\partial XY} - \frac{\partial \Psi}{\partial X}\frac{\partial^2 \Psi}{\partial Y^2} = \nu_\circ \frac{\partial}{\partial Y} C \frac{\partial^2 \Psi}{\partial Y^2} \tag{661}$$

$$\frac{\partial \Psi}{\partial Y}\frac{\partial h}{\partial X} - \frac{\partial \Psi}{\partial X}\frac{\partial h}{\partial Y} = \nu_\circ \frac{\partial}{\partial Y}\left(\frac{C}{Pr}\frac{\partial h}{\partial Y}\right) + \nu_\circ C \left(\frac{\partial^2 \Psi}{\partial Y^2}\right)^2 \tag{662}$$

Similarity variable

$$\eta = \frac{y}{\delta(x)} \qquad \delta = \sqrt{\frac{2\nu_\circ x}{U}} \tag{663}$$

Streamfunction ansatz for zero pressure gradient

$$\Psi = U\delta f(\eta) \qquad h = h_\circ g(\eta) \tag{664}$$

Similarity function equations

$$
\begin{aligned}
(Cf'')' + ff'' &= 0 &(665)\\
(\frac{C}{Pr}g')' + fg' &= -CEc\,(f'')^2 &(666)
\end{aligned}
$$

where the Eckert number is

$$Ec = \frac{U^2}{h_\circ} = (\gamma - 1)M^2 \quad \text{for perfect gases} \qquad (667)$$

Transport property approximation

$$C = 1 \quad \rho\mu = \rho_\circ\mu_\circ \qquad Pr = \frac{c_P\mu}{k} = \text{constant} \qquad (668)$$

Approximate equation set:

$$
\begin{aligned}
f''' + ff'' &= 0 &(669)\\
g'' + Prfg' &= -PrEc\,(f'')^2 &(670)
\end{aligned}
$$

5.8.2 Energy Equation

Integration of the energy equation results in the integral relationship for heat flux at the wall

$$q_w = \frac{\partial}{\partial x}\left(\rho_e u_e h_{t,e}\Theta_h\right) \qquad \Theta_h = \int_0^\infty \frac{\rho u}{\rho_e u_e}\left(\frac{h_t}{h_{t,e}} - 1\right) \mathrm{d}y \qquad (671)$$

where Θ_h is the *energy thickness*.

The recovery factor r determines the wall enthalpy in adiabatic flow,

$$h_r = h_w(q_w = 0) = h_\infty + r\frac{1}{2}u_\infty^2 \qquad (672)$$

The recovery factor is found to be an increasing function of the Prandtl number. In gases,

$$
\begin{aligned}
&r \simeq Pr^{1/2} \qquad \text{laminar boundary layers}\\
&r \simeq Pr^{1/3} \qquad \text{turbulent boundary layers}
\end{aligned}
\qquad (673)
$$

Unity Prandtl Number For $Pr = 1$, the energy equation is

$$\rho u\frac{\partial h_t}{\partial x} + \rho v\frac{\partial h_t}{\partial y} = \frac{\partial}{\partial y}\left(\mu\frac{\partial h_t}{\partial y}\right) \qquad h_t = h_e + \frac{u_e^2}{2} \qquad (674)$$

This has as a solution in adiabatic flow

$$h_t = h_e + \frac{u_e^2}{2} = h_\infty + \frac{u_\infty^2}{2} = \quad \text{constant for} \quad q_w = 0 \qquad (675)$$

Therefore, the recovery enthalpy is

$$h_r = h_\infty + \frac{u_\infty^2}{2} \tag{676}$$

From the exact correspondence to the x-momentum equation, the general, $q_w \neq 0$, solution is $h_t = a + bu$. This leads directly to the *Crocco* integral

$$h = h_\infty + [h_w - h_r]\left(1 - \frac{u}{u_\infty}\right) + \frac{u_\infty^2}{2}\left(1 - \frac{u^2}{u_\infty^2}\right) \tag{677}$$

The Stanton number can be derived from this result in the form of Reynolds analogy

$$St = \frac{C_f}{2} \tag{678}$$

The generalization of this to other Prandtl numbers that is valid for laminar and turbulent boundary layers in gases is

$$St \simeq \frac{C_f}{2Pr^{2/3}} \tag{679}$$

General Prandtl Number For similarity solutions, the nondimensional enthalpy can be found by integration of the energy equation, simplest when $C = 1$, and $Pr =$ constant.

$$g'' + Pr\, f g' = -Pr\, Ec\, (f'')^2 \tag{680}$$

This equation can be integrated exactly to yield

$$g(\eta) = g(0) + g'(0)\int_0^\eta F(\eta'; Pr)\,\mathrm{d}\eta' - Pr\, Ec \int_0^\eta F(\eta'; Pr)\left[\int_0^{\eta'} \frac{(f''(\xi))^2\,\mathrm{d}\xi}{F(\xi; Pr)}\right]\mathrm{d}\eta' \tag{681}$$

where

$$F(\eta; Pr) = \int_0^\eta \exp\left(-Pr\int_0^{\eta'} f(\xi)\,\mathrm{d}\xi\right)\mathrm{d}\eta' \tag{682}$$

and the boundary conditions yield

$$g(0) = \frac{h_w}{h_e} \qquad q_w = -k\left.\frac{\partial T}{\partial y}\right|_w = -\frac{k}{c_P}\frac{h_e}{\delta}\frac{\rho}{\rho_e}g'(0) \tag{683}$$

This results in a recovery factor of

$$r = 2Pr\int_0^\infty F(\eta; Pr)\left[\int_0^\eta \frac{(f''(\xi))^2\,\mathrm{d}\xi}{F(\xi; Pr)}\right]\mathrm{d}\eta \tag{684}$$

which for a laminar flat plate boundary layer has the approximate value

$$r \approx Pr^{1/2} \qquad 0.1 \leq Pr \leq 3.0 \tag{685}$$

The Stanton number is

$$St \equiv \frac{q_w}{\rho_e u_e (h_w - h_r)} = \frac{G(Pr)}{\sqrt{2} Pr Re_x^{1/2}} \qquad G(Pr) = \left[\int_0^\infty F(\eta; Pr)\, \mathrm{d}\eta \right]^{-1} \tag{686}$$

and for a flat plate boundary layer G can be approximated as

$$G \approx 0.4969 Pr^{1/3} \qquad 0.1 \leq Pr \leq 3.0 \tag{687}$$

so that the Stanton number for flat plate gas flow is approximately

$$St = \frac{0.33206}{Pr^{2/3} Re_x^{1/2}} \tag{688}$$

Coordinate stretching The physical coordinate can be computed from the transformed similarity variable and the velocity profile

$$y \sqrt{\frac{u_\infty}{2\nu_\infty x}} = \int_0^\eta \frac{\rho_\infty}{\rho}\, \mathrm{d}\eta \tag{689}$$

The density profile can be computed from the temperature profile since the pressure is constant across the boundary layer. For an ideal gas

$$\frac{\rho_\infty}{\rho} = \frac{T}{T_\infty} \tag{690}$$

For the case of $Pr = 1$ and a perfect gas, the temperature profile is

$$\frac{T}{T_\infty} = 1 + \frac{\gamma - 1}{2} M_\infty^2 \int_0^\eta \left(1 - f'^2\right) \mathrm{d}\eta' \tag{691}$$

where $u = u_\infty f'(\eta)$. The coordinate transformation is then

$$y \sqrt{\frac{u_\infty}{2\nu_\infty x}} = \eta + \frac{\gamma - 1}{2} M_\infty^2 \int_0^\eta \left(1 - f'^2\right) \mathrm{d}\eta' \tag{692}$$

If we suppose that the viscosity varies as $\mu \sim T^\omega$, then the skin friction coefficient is

$$C_f = \frac{\sqrt{2} f''(0)}{Re_x^{1/2}} \frac{1}{\left(1 + \frac{\gamma-1}{2} M_\infty^2\right)^{1-\omega}} \tag{693}$$

5.8.3 Moving Shock Waves

For a moving shock wave, the boundary conditions in the shock fixed frame are that the wall is moving with the upstream velocity w_1 and the freestream condition is w_2. If the reference velocity is w_2, then boundary conditions on f are

$$f(0) = 0, \quad f'(0) = \frac{u_w}{u_e} = \frac{w_1}{w_2} \qquad f'(\infty) = 1 \tag{694}$$

This results in a *negative displacement thickness.*

5.8.4 Weak Shock Wave Structure

In contrast to the usual Boundary layer equations, here $\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0$, and only derivatives in the x direction are considered.

$$\rho u = \rho_1 u_1 \tag{695}$$

$$P + \rho_1 u_1 u - \frac{4}{3}\mu' \frac{\partial u}{\partial x} = P_1 + \rho_1 u_1^2 \tag{696}$$

$$h + \frac{u^2}{2} - \frac{4}{3}\frac{\mu'}{\rho_1 u_1} u \frac{\partial u}{\partial x} - \frac{k}{\rho_1 u_1}\frac{\partial T}{\partial x} = h_1 + \frac{u_1^2}{2} \tag{697}$$

where

$$\mu' = \mu + \frac{3}{4}\mu_v \tag{698}$$

Entropy creation by gradients:

$$s_2 - s_1 = \frac{1}{\rho_1 u_1}\int_{-\infty}^{+\infty} \left[\frac{4}{3}\frac{\mu'}{T}\left(\frac{\partial u}{\partial x}\right)^2 + k\left(\frac{1}{T}\frac{\partial T}{\partial x}\right)^2\right] \mathrm{d}x \tag{699}$$

Weak shock thickness estimate based on maximum slope:

$$\Delta_m = \frac{8\mu'}{3\rho c}\frac{1}{M_{1n} - 1} \qquad \mu' = \mu + \frac{3}{4}\mu_v \tag{700}$$

For a perfect gas (γ = constant), the mean free path can be estimated as

$$\Lambda = \left(\frac{\pi\gamma}{2}\right)^{1/2} \frac{\mu}{\rho c} \tag{701}$$

and the shock thickness for $\gamma = 1.4$, $\mu_v = 0$, is

$$\Delta_m = \frac{1.8\Lambda}{(M_{1n} - 1)} \tag{702}$$

5.9 Creeping Flow

In the limit of zero inertia, the flow is described by *Stokes* approximation to the momentum equation

$$\boldsymbol{\nabla} P = \boldsymbol{\nabla} \cdot \boldsymbol{\tau} \tag{703}$$

If the viscosity and density are constant this is equivalent to

$$\boldsymbol{\nabla} P = \mu \nabla^2 \mathbf{u} \qquad \text{or} \qquad \boldsymbol{\nabla} P = -\mu \boldsymbol{\nabla} \times \boldsymbol{\omega} \tag{704}$$

Applying the divergence and curl operations to these equations yields

$$\nabla^2 P = 0 \qquad \text{or} \qquad \nabla^2 \boldsymbol{\omega} = 0 \tag{705}$$

The Reynolds number enters solely through the boundary conditions. Consider a flow with characteristic velocity U, lateral dimension L and viscosity μ. If the velocity is specified at the boundaries,

$$\mathbf{u} = U g(\mathbf{x}/L, \text{geometry}) \tag{706}$$

then the pressure distribution can be obtained by integrating the momentum equation to get

$$P = \frac{\rho U^2}{Re_L} f(\mathbf{x}/L, \text{geometry}) \qquad Re_L = \frac{\rho U L}{\mu} \tag{707}$$

If the pressure is specified at the boundaries,

$$P = \rho U^2 f(\mathbf{x}/L, \text{geometry}) \tag{708}$$

then the velocity will be given by

$$\mathbf{u} = U Re_L g(\mathbf{x}/L, \text{geometry}) \tag{709}$$

For flows in two space dimensions, a streamfunction ψ can be used to satisfy the continuity equation. In cartesian coordinates, the streamfunction for Stokes flow of a constant viscosity fluid will satisfy the *Biharmonic* equation

$$\nabla^4 \psi = 0 \tag{710}$$

Stokes Sphere Flow The force on a moving body in viscous flow is

$$\mathbf{F} = \int_{\partial\Omega} \boldsymbol{\tau} \cdot \hat{\mathbf{n}} \, \mathrm{d}A - \int_{\partial\Omega} P \hat{\mathbf{n}} \, \mathrm{d}A \tag{711}$$

Estimating the magnitude of the integrals, the force in a particular direction will have the magnitude

$$F = C \mu U L \tag{712}$$

The constant C will in general depend on the shape of the body, the direction $\hat{\mathbf{x}}$ of the force and the motion of the body.

For a sphere, the flow can be solved by using Stokes axisymmetric streamfunction ψ. The velocity components are:

$$u_r = \frac{1}{r^2 \sin\phi} \frac{\partial \psi}{\partial \phi} \quad (713)$$

$$u_\phi = -\frac{1}{r \sin\phi} \frac{\partial \psi}{\partial r} \quad (714)$$

The analog of the biharmonic equation is

$$\left[\frac{\partial^2}{\partial r^2} + \frac{\sin\phi}{r^2} \frac{\partial}{\partial \phi} \left(\frac{1}{\sin\phi} \frac{\partial}{\partial \phi} \right) \right]^2 \psi = 0 \quad (715)$$

The boundary conditions at the surface of the sphere are:

$$\psi = 0 \qquad \frac{\partial \psi}{\partial r} = 0 \qquad \frac{\partial \psi}{\partial \phi} = 0 \qquad r = a \quad (716)$$

and the flow approaches a uniform flow far from the sphere

$$\lim_{r \to \infty} \psi = \frac{U r^2}{2} \sin^2 \phi \quad (717)$$

The solution is

$$\psi = \frac{U}{4} a^2 \sin^2 \phi \left(\frac{a}{r} - \frac{3r}{a} + \frac{2r^2}{a^2} \right) \quad (718)$$

The pressure on the body is found by integrating the momentum equation

$$P = P_\infty - \frac{3\mu a U}{2r^2} \cos\phi \quad (719)$$

and the force (drag) is directed opposite to the direction of motion of the sphere with magnitude

$$D = 6\pi\mu U a \qquad C_D \equiv \frac{D}{1/2 \rho U^2 \pi a^2} = \frac{24}{Re} \quad Re = \frac{\rho U 2a}{\mu} \quad (720)$$

For a thin disk of radius a moving normal to the freestream the drag is

$$D = 16\pi\mu U a \quad (721)$$

and moving parallel to the freestream

$$D = \frac{32}{3} \mu U a \quad (722)$$

Oseen's Approximation The inertial terms neglected in Stokes' approximation become significant at a distance $r \sim a/Re$. The *Oseen equations* are a uniform approximation for incompressible viscous flow over a body. If the mean flow at large distances from body is U in direction x, then the Oseen equations are:

$$\nabla \cdot \mathbf{u} = 0 \tag{723}$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho U \frac{\partial \mathbf{u}}{\partial x} = -\nabla P + \mu \nabla^2 \mathbf{u} \tag{724}$$

This results in a corrected drag law (the flow now has a wake) for the sphere

$$C_D = \frac{24}{Re}\left(1 + \frac{3Re}{16} + \frac{9}{160} Re^2 \ln Re + \ldots\right) \tag{725}$$

Reynolds Lubrication Theory Incompressible flow in a two-dimensional channel with a slowly-varying height $h(x)$ and length L can be treated as a "boundary layer"-like flow if

$$\frac{L}{h}\frac{\partial h}{\partial x} \ll 1 \qquad \text{which implies that} \qquad v \approx u \frac{\partial h}{\partial x} \tag{726}$$

The thin-layer or lubrication equations result when the channel is very thin $h/L \to 0$, and viscous forces dominate inertia $Re \ll 1$.

$$\frac{\partial \rho h}{\partial t} + \frac{\partial \rho h u}{\partial x} = 0 \tag{727}$$

$$0 = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial y} \mu \frac{\partial u}{\partial y} \tag{728}$$

$$0 = -\frac{\partial P}{\partial y} \tag{729}$$

For a constant property flow, the velocity is given at any point in the channel by the Couette-Poiseuille expression of parallel flow if the lower boundary is moving with velocity U and the upper boundary is at most moving in the y direction

$$u = -\frac{h^2}{2\mu}\frac{\partial P}{\partial x}\frac{y}{h}\left(1 - \frac{y}{h}\right) + U\left(1 - \frac{y}{h}\right) \tag{730}$$

Combining this result with the continuity equation yields the *Reynolds* lubrication equation

$$\frac{1}{\mu}\frac{\partial}{\partial x}\left(h^3 \frac{\partial P}{\partial x}\right) = 6U \frac{\partial h}{\partial x} + 12 \frac{\partial h}{\partial t} \tag{731}$$

For a *slipper pad bearing*, the pressure is equal to the ambient value $P_\circ$ at $x = 0$ and $x = L$ and the gap height h is

$$h = h_\circ \left(1 - \alpha \frac{x}{L}\right) \qquad \alpha \ll 1 \tag{732}$$

The pressure is given by

$$P - P_\circ = \frac{\mu U L}{\alpha h_\circ^2}\left[6\left(\frac{h_\circ}{h} - 1\right) - 3\frac{h^*}{h_\circ}\left(\frac{h_\circ^2}{h^2} - 1\right)\right] \tag{733}$$

where h^* is the gap height at the location of the pressure maximum

$$\frac{h^*}{h_\circ} = 2\frac{1-\alpha}{2-\alpha} \approx 1 - \frac{\alpha}{2} - \frac{\alpha^2}{4} + O(\alpha^3) \tag{734}$$

and the maximum pressure is approximately

$$P_{max} - P_\circ \approx \frac{3}{4}\alpha\frac{\mu U L}{h_\circ^2} + O(\alpha^2) \tag{735}$$