Appendix C

Perfect Gas Analytical Solutions

The perfect gas has a constant heat capacity and we assume a fixed composition across the shock, so that for both upstream and downstream states, the equation of state is given by

$$P = \rho R T \tag{C.1}$$

$$h = c_P T \tag{C.2}$$

The classical studies of gas dynamics use this model extensively since the jump conditions and many other problems can be solved exactly. A compendium of exact solutions for perfect gases is given in **?**.

C.1 Incident Shock Waves

The standard approach in classical gas dynamics is to express the solutions in terms of nondimensional variables and parameters. Instead of the specific heat capacity, the gas is characterized by the nondimensional parameter $\gamma = c_P/c_v$, the ratio of specific heats. Instead of velocities, the Mach number is used

$$M = \mathrm{w}/a \tag{C.3}$$

For a perfect gas, because the specific heat is constant, there is a single sound speed.

$$a = \sqrt{\gamma R T} \tag{C.4}$$

The conservation relationships can be analytically solved in terms of the jump or change in properties,

$$[F] = F_2 - F_1 \ , \tag{C.5}$$

across the wave

$$\frac{[P]}{P_1} = \frac{2\gamma}{\gamma+1}\left(M_1^2 - 1\right) \tag{C.6}$$

$$\frac{[\mathrm{w}]}{a_1} = -\frac{2}{\gamma+1}\left(M_1 - \frac{1}{M_1}\right) \tag{C.7}$$

$$\frac{[v]}{v_1} = -\frac{2}{\gamma+1}\left(1 - \frac{1}{M_1^2}\right) \tag{C.8}$$

$$\frac{[s]}{R} = -\ln\left(\frac{P_{t2}}{P_{t1}}\right) \tag{C.9}$$

$$\frac{P_{t2}}{P_{t1}} = \frac{1}{\left(\frac{2\gamma}{\gamma+1}M_1^2 - \frac{\gamma-1}{\gamma+1}\right)^{\frac{1}{\gamma-1}}}\left(\frac{\frac{\gamma+1}{2}M_1^2}{1 + \frac{\gamma-1}{2}M_1^2}\right)^{\frac{\gamma}{\gamma-1}} \tag{C.10}$$

Using the transformation from wave-fixed to laboratory frame, we have

$$[\mathrm{w}] = -[\mathrm{u}] \tag{C.11}$$

so that

$$\frac{[\mathrm{u}]}{a_1} = \frac{2}{\gamma+1}\left(M_1 - \frac{1}{M_1}\right) \tag{C.12}$$

$$\tag{C.13}$$

We can also analytically express the shock adiabat or Hugoniot

$$\frac{P_2}{P_1} = \frac{\frac{\gamma+1}{\gamma-1} - \frac{v_2}{v_1}}{\frac{\gamma+1}{\gamma-1}\frac{v_2}{v_1} - 1} \tag{C.14}$$

or alternatively

$$\frac{P_2}{P_1} = 1 + \frac{2\gamma}{\gamma+1}\left(M_1^2 - 1\right) \tag{C.15}$$

$$= \frac{2\gamma}{\gamma+1}M_1^2 - \frac{\gamma-1}{\gamma+1} \tag{C.16}$$

$$\frac{\rho_2}{\rho_1} = \frac{\gamma+1}{\gamma - 1 + \frac{2}{M_1^2}} \tag{C.17}$$

$$M_2^2 = \frac{M_1^2 + \frac{2}{\gamma-1}}{\frac{2\gamma}{\gamma-1}M_1^2 - 1} \tag{C.18}$$

Another useful equation is *Prandtl's relation*,

$$\mathrm{w}_1\mathrm{w}_2 = a^{*2}, \tag{C.19}$$

where a^* is the sound speed at a sonic point obtained in a fictitious isentropic process in the upstream flow.

$$a^* = \sqrt{2\frac{\gamma-1}{\gamma+1}h_t}, \qquad h_t = h + \frac{\mathrm{w}^2}{2} \tag{C.20}$$

C.2 Reflected Shock Waves

Several relationships for reflected waves can be derived by based on the fact that fluid adjacent to a stationary surface must be stationary. Figure 7.8 (Section 7.6) illustrates a possible geometry for wave reflection. The above condition requires that

$$\mathrm{u}_1 = \mathrm{u}_3 = 0\ . \tag{C.21}$$

Therefore, the jump in velocity across the reflected wave,

$$[\mathrm{u}]_R = \mathrm{u}_3 - \mathrm{u}_2 = -\mathrm{u}_2 \tag{C.22}$$

is the exact opposite of the jump in velocity across the incident wave,

$$[\mathrm{u}]_I = \mathrm{u}_2 - \mathrm{u}_1 = \mathrm{u}_2\ , \tag{C.23}$$

or

$$[\mathrm{u}]_I = -[\mathrm{u}]_R \tag{C.24}$$

The Rayleigh line equation (7.16) can be expressed in terms of jumps in properties, i.e.

$$[\mathrm{u}]^2 = -[P][v] \tag{C.25}$$

Now we relate the Rayleigh line of each wave

$$[P]_R[v]_R = [P]_I[v]_I \ . \tag{C.26}$$

Pressure Jump

Using the perfect gas Hugoniot relationship (C.14) for both the incident and reflected waves, we can eliminate the volume jumps and find a relationship between the pressure ratios across the incident and reflected waves. Using the notation in Section 7.6,

$$\frac{P_3}{P_2} = \frac{(3\gamma - 1)\dfrac{P_2}{P_1} - (\gamma - 1)}{(\gamma - 1)\dfrac{P_2}{P_1} + (\gamma + 1)} \tag{C.27}$$

The pressure ratio across the reflected shock is always less than across the incident shock and has a limiting value for large incident shock speeds of

$$\frac{P_3}{P_2} \rightarrow \frac{3\gamma - 1}{\gamma - 1} \quad \text{as} \quad \frac{P_2}{P_1} \rightarrow \infty \tag{C.28}$$

On the other hand, for small incident shock speeds, the pressure ratio across the reflected and incident shock waves approaches 1. In this limit, if we expand about the initial state,

$$\frac{P_3}{P_2} - 1 = \frac{P_2}{P_1} - 1 - \frac{\gamma - 1}{2\gamma}\left(\frac{P_2}{P_1} - 1\right)^2 + \left(\frac{\gamma - 1}{2\gamma}\right)^2\left(\frac{P_2}{P_1} - 1\right)^3 + \dots \ , \tag{C.29}$$

and retain only the first order terms of the series, we obtain the acoustic result, i.e. the pressure rise across the reflected shock is equal to the rise across the incident shock. In other words, the total pressure rise $(P_3 - P_1)$ is twice the pressure rise due to the incident wave $(P_2 - P_1)$.

$$P_3 - P_1 \approx 2\,(P_2 - P_1) \quad \text{for acoustic waves} \tag{C.30}$$

Mach Number

Similarly, we can determine an expression for the reflected shock Mach number. First, we define the incident and reflected shock Mach numbers.

$$M_I = \frac{\mathrm{U}_I}{a_1} \tag{C.31}$$

$$M_R = \frac{\mathrm{U}_R + \mathrm{u}_2}{a_2} \ . \tag{C.32}$$

Then, using the velocity jump relation (C.12) and recalling (C.21), we relate the two Mach numbers

$$M_R - \frac{1}{M_R} = \frac{a_1}{a_2}\left(M_I - \frac{1}{M_I}\right) \ . \tag{C.33}$$

The left-hand side is a function α of the incident shock speed

$$\alpha = \frac{a_1}{a_2}\left(M_I - \frac{1}{M_I}\right) . \tag{C.34}$$

For a specified incident shock Mach number, we can compute α and find the reflected shock Mach number by solving the resulting quadratic equation

$$M_R = \frac{\alpha + \sqrt{\alpha^2 + 4}}{2} . \tag{C.35}$$

From the incident shock jump conditions, α ranges between zero and a maximum value which is only a function of γ. Taking the limit as $M_I \to \infty$, we find that

$$\alpha_{max} = \frac{\gamma + 1}{\sqrt{2\gamma(\gamma - 1)}} \tag{C.36}$$

which means that the reflected shock Mach number ranges betweem one and a maximum value of

$$M_{R,max} = \sqrt{\frac{2\gamma}{\gamma - 1}} . \tag{C.37}$$

Enthalpy

For strong incident shock waves, we can derive from the reflected shock relationships (7.46)-(7.48), the approximate results

$$h_2 \approx h_1 + \frac{1}{2}U_I^2 \tag{C.38}$$

$$h_3 \approx h_2 + \frac{1}{2}U_I^2 \tag{C.39}$$

so that the enthalpy behind a strong reflected shock wave is

$$h_3 \approx h_1 + U_I^2 \tag{C.40}$$

which is very useful in estimations of the reservoir enthalpy in the reflected shock tunnels.